THE
TEACHING OF ARITHMETIC

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IN THE INFANT AND JUNIOR SCHOOL

BY

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THE "WELBENT" NUMBER SCHEME "THREE TERM
NATURE STUDY" ETC.

GEORGE G. HARRAP & COMPANY LTD.
LONDON SYDNEY TORONTO BOMBAY

First published 1928
by George G. Harrap & Co. Ltd.
182 *High Holborn, London, W.C.*1

Reprinted: February 1932; *January* 1940;
April 1948

THIS REPRINT IS IN CONFORMITY WITH
THE AUTHORIZED ECONOMY STANDARDS

MADE IN GREAT BRITAIN. PRINTED AT THE PITMAN PRESS, BATH

PREFACE

THIS book is intended for students in training and for teachers of children under ten years of age.

It has been written in response to a felt need for a book dealing with such practical individual work in arithmetic as has been found to succeed so well with small children. The first noteworthy book dealing with this stage of arithmetic was Miss Punnett's *The Groundwork of Arithmetic.* However, the author emphasized class teaching rather than individual or group work, and though many books full of suggestions have appeared since, there are few that deal exhaustively with all fundamental processes.

Few teachers will agree on the exact order in which the different processes are to be taught, and a variety of text-books will be in use by the children, so I have deemed it best to take each subject in turn and deal with it exhaustively. It is hoped that this will make the book useful to teachers who form their own syllabuses. Suggestions for a scheme are given in an appendix.

In order to guide students in their reading, and to meet the requirements of such examinations as the Froebel, the usual questions (*e.g.*, formal training, cultural value of arithmetic, etc.) are touched upon. Quotations and references are introduced freely in order to encourage the student to read standard authors on the various subjects.

One of the characteristics of the book is the large number of illustrations. While no one will find it necessary to have all the cards figured, it is hoped that these will give ideas to teachers who make their own apparatus and will suggest exercises for individual children who have been found weak on some point. In the chapters on each rule apparatus is recommended and certain fundamental requirements are discussed.

A list of apparatus, with publishers' names, is given in Appendix I.

Though individual work is stressed throughout, class-work and a certain amount of drill are considered essential, and are therefore included.

A whole chapter has been devoted to number-study and chart-work, for I am convinced that children have a natural interest in numbers as numbers, and that this interest is being stifled in some schools by over-emphasis on the concrete.

Memory-work is extremely important, as without it no speed can be acquired. For children under ten accuracy is more important than speed, but the latter *must be prepared for*. The use of charts is a most economical way of memorizing various types of facts and of acquiring facility in locating and manipulating numbers.

There should be two parts to every arithmetic scheme for small children. The first is the " working syllabus." This is carefully graded, and, in the standards, may be represented by some arithmetic text-book. Every child is required to understand and show facility in dealing with the processes involved in the examples set. In fact, this part of the work can and should be tested in regard to method, accuracy, and speed.

The second part consists of exercises deliberately set to give the child experiences which will prepare him for future and more advanced work. The full value of these exercises will be appreciated only by the teachers in the higher classes, for in the lower classes, though they increase a child's outlook, they do not improve his skill appreciably. They cannot be tested or measured, and for this reason are less satisfying to a teacher! It is in elementary schools, where the home surroundings of the children are not always conducive to thought and to invention, that these exercises are so important. To be of value they must be carefully thought out, and each must have some idea or aspect of its own.

My thanks are due to Miss Cummins, of Dowanhill Training College, for her untiring help with the diagrams, all of which she sketched for me; also to the teachers in St Charles's Demonstration School for their constant interest and their careful testing of the various cards.

PREFACE

The titles of the chief works to which I referred when writing this book are given in the bibliography.

Thanks are due to the following publishers and authors for permission to quote from their books : Messrs Macmillan and Co., Ltd., for three poems from *Sing-Song*, by Christina Rossetti; the Macmillan Company, New York, for extracts from *A Text-book in the Principles of Science Teaching*, by G. R. Twiss, and *The Teaching of Elementary Mathematics*, by D. E. Smith ; Sir Isaac Pitman and Sons, Ltd., for an extract from *The Teaching of Arithmetic*, by F. F. Potter ; Messrs Edward Arnold and Co., for an extraet from *The Dawn of Mind*, by Margaret Drummond; Dr P. B. Ballard, Messrs Hodder and Stoughton, Ltd., and the London University Press, for extracts from *Mental Tests* and *Fundamental Arithmetic*, by P. B. Ballard ; Messrs Methuen and Co., Ltd., for an extract from *The Psychology of Education*, by Kennedy Fraser; the University Tutorial Press, for an extract from *Child Mind*, by Benjamin Dumville; Dr Maria Montessori and Messrs William Heinemann, Ltd., for extracts from *The Montessori Method* and *The Advanced Montessori Method*, by Dr Maria Montessori ; Mr John Murray, for an extract from *British Weights and Measures*, by Sir Charles M. Watson. I am also indebted to Messrs E. J. Arnold and Son, Ltd., for permission to reproduce apparatus in the " Welbent " Series, to Messrs Philip and Tacey, Ltd., for permission to reproduce diagrams of Montessori apparatus and of the " Riverside " Money Charts, and to Messrs Evans Brothers for permission to use matter and diagrams contained in six articles written for *Child Education* in 1925.

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in solving a problem in greatest common divisor may show themselves years later in commerce, in banking, or in one of the learned professions. Hence arithmetic, when taught with this in mind, gives to the pupil not knowledge of facts alone, but that which transcends such knowledge, namely, power.[1]

It is not here intended to advocate the doctrine of formal training. It is fully realized that a 'spread' or 'carrying' or 'flowing over' of improvement from one subject to another depends on the subjects having some elements in common, and that it is a mistake to give children obsolete, complicated, lengthy, and altogether unreal examples simply for the sake of training. But it is contended that arithmetic provides exercises in reasoning, and that without exercise the mind loses power—that concentrated work in arithmetic does produce a general improvement in the child's work, due probably, as Meumann points out, to the acquisition of certain habits—"attention, favourable general attitude, suppression of superfluous movements or tension, use of appropriate imagery, acquisition of uniform emotional disposition, avoidance of mental uneasiness, acquisition of greater self-confidence."[2]

The exercises in reasoning afforded by arithmetic can have their full value or force only when the *child himself* investigates and draws inferences—when he fully realizes the conditions of the problem, and is able to search for ways and means of solving it. It is, in fact, the method that is important—not the facts learnt, but the way in which they are learnt.

But there is still another side to this disciplinary value. Arithmetic involves not a small amount of drudgery—tables must be learnt, speed must be acquired. Ten minutes of concentrated work every day on this part of the subject does much towards maturing and steadying the child. The nature of this drill is discussed later.

All this work in reasoning out problems and going through more or less mechanical exercises has its own very tangible reward. In dealing with life, in nature-study, in history, in geography, the child is constantly coming up against excep-

[1] Smith, *The Teaching of Elementary Mathematics* (Macmillan Co., New York).

[2] Rusk, *Introduction to Experimental Education.* (In this connexion read *The Teaching of Arithmetic*, by Potter, pp. 118 and 119.)

tions to rules. He is constantly being taught not to generalize. This is a lesson he *must* learn, yet he finds it hard, for he is not mature enough to appreciate its beauty; exceptions constantly falsify his reasonings, he feels he can never be quite sure of his premises, and he comes to doubt himself. Arithmetic is free from these perplexing exceptions; hence it is satisfying to a child's craving for the definite—the 'yes' or 'no,' the 'right' or 'wrong.'

(2) Besides the disciplinary value arithmetic has another, *cultural value.* This lies in its connexion with man's life, with his physical environment, with the development of his occupations, and with his advance in science. Astronomy, physics, mechanics, engineering, aeroplane- and ship-building, architecture—all depend on mathematics. The wonderful precision as to time and place with which a solar eclipse can now be predicted is a triumph of mathematics. Teachers sometimes overlook the very human interest of mathematics and regard it as a dry, abstract subject. A little study of the history of mathematics, of the development of occupations, and of the progress of the sciences would soon disillusion them and show how the subject teems with interest.

We must take care that the children get a chance of appreciating this aspect of arithmetic by bringing it into connexion with their lives and interests. This we can do by encouraging them to investigate, to construct, to experiment along any line that interests them at the moment.

Section II. The Child's Attitude towards Arithmetic

The following quotation is interesting because it shows that for all types of mind concrete material is necessary in the early stages of number-work.

> Some children are 'thing thinkers.' They care to think, and can think successfully, only in terms of what they can see, hear, touch, and manipulate. Words and symbols have meaning to such children only in the presence of the things themselves which the words and symbols represent, and their main interests are in doing, bringing things to pass, getting practical results. Others are 'idea thinkers.' These grasp ideas much more easily than do

those of the former group, and it is comparatively easy to interest them in abstractions. At the higher school age, many of them are attracted by the broader principles and comprehensive statements, but are not fond of details. They like talking better than doing. As Professor J. E. Miller has said, many of them have an almost fatal facility in handling abstract formulæ and juggling with symbols. The teacher must constantly be adjusting his methods to provide for both of these so-called types of mind ; but fortunately the greatest need in both types is the same. Both require to have constantly kept before them an abundance of concrete facts and materials for observation, models, maps, specimens, simple apparatus. With such material before them the 'thing thinkers' can be assisted in the processes of discrimination, comparison, abstraction, generalization, and the coupling in memory of the things with the symbols that stand for them in thought. With the same means, the 'idea thinkers' can be taught to avoid premature wordy flights by compelling them to square what they say with what they see. Thus both types will get practice in the kinds of mental operations in which they are deficient.[1]

In order to treat adequately of the child's attitude towards arithmetic it would be necessary to discuss the origin of number concepts and the stages of development of the child, together with the characteristic interests and activities of each stage.[2] Here, however, it is possible only to give a brief discussion of some practical questions.

All children would love arithmetic if they were properly taught. Failure in arithmetic is almost always due to wrong methods or to a want of understanding of a child's mind. Children are busy people—they like movement, they love puzzles, they revel in rhythm, and they are full of interest in life. All these cravings can have an outlet in arithmetic —it is merely a matter of using each child's gifts in the best way. No one who has watched children working in a good infants' school can think that the child is the obstacle to good results in arithmetic. One sees mites of five and six so wrapped up in their work as to be oblivious of all that is going on around them. What, then, is the cause of so many children's failing to reach an average standard in arithmetic ?

[1] Twiss, *A Text-book in the Principles of Science Teaching* (Macmillan Co., New York).

[2] In this connexion read *The Psychology of Number*, by McLellan and Dewey, *The Psychology of the Common Branches*, by Freeman, and *Introduction to Experimental Education*, by Rusk, p. 280.

After questioning many grown-ups who own to not having succeeded in arithmetic at school, it seems that discouragement and fear are two important causes of unsuccess. These being within the child, the unobservant teacher is quite unaware of them. This discouragement shows itself in various ways. We have the child who says " I hate arithmetic "—which, being interpreted, means " I can't do arithmetic." The attitude taken up is the very natural one of self-defence. Again, we have the child who says nothing, but who sedulously avoids arithmetic. Such a child has sometimes never attempted much in arithmetic, but he has realized that the other children around him have grasped what he has not—he does not quite know what it all means—so he wisely shuns it.

In no subject is self-confidence more necessary than in arithmetic, and in no subject is it so easily killed, for, alas! sums must be either right or wrong. " Nothing succeeds like success " is specially true of a child's work in arithmetic. It is deplorable to find children of eight or nine already painfully aware of their limitations, yet this is not unusual, and the worst of it is that this self-diffidence multiplies the child's difficulties and increases his limitations—in fact, it *generates stupidity*. The teacher must see to it that all the children succeed in their measure.

The chief external causes of unsuccess—which, of course, act on and produce the discouragement and fear of which we have just been speaking—seem to be

(1) Hurry in the initial stages.
(2) Application of unsuitable stimuli.
(3) Lack of suitable apparatus.
(4) Faults in organization and method.

(1) Hurry in the Initial Stages. A small child ought never to be hurried while he is calculating. He may annoy the teacher by continuing to count every bead on his bar again and again, but if he does this it is a sure sign that it is necessary for his confidence in his work. A small boy of five morning after morning for over three weeks went straight to the case containing the " Welbent " cards and took one with a picture of a ship and worked through the five sums on it.

He got them right each time, and then proceeded to more advanced work. It seemed that this favourite exercise gave him a feeling of satisfaction and confidence for his other work. Suddenly one day he did not take the card; he had outgrown it, and it ceased to satisfy him. Similarly, a small girl of five could not settle to work in the morning without first counting all the beads on the horizontal bead-bar which hung on the wall. For weeks she counted them in ones up to thirty, then suddenly one day she counted in twos, and finally in tens.

(2) **Application of Unsuitable Stimuli.** In Section IV different stimuli are recommended and briefly discussed. In using stimuli we must bear in mind that we have no right to pit the strength of a weak child against that of a brighter child, even though they be of the same age. This causes a sense of inferiority to develop, which inhibits the child's best efforts in the subject. Some little people develop very slowly, and we must leave them to do so quietly. If we hustle them they lose their nerve and their self-confidence, and it will then be difficult ever to make them concentrate on an arithmetic exercise.

We may say that all competition and punishments are positively bad for children under eight. For children between eight and twelve competitions are useful under special conditions, to be discussed in Section IV.

(3) **Lack of Suitable Apparatus.** In the quotation at the beginning of this section it is pointed out how necessary it is for a child to have concrete examples. A great many failures in arithmetic are due to lack of the necessary fundamental ideas, which can be acquired only through concrete experience. Yet there must not be too great a variety of apparatus for computation. In the initial stages it is necessary to let a child count a great variety of objects in order to avoid his thinking that the number names refer only to beads or counters, but when he is doing formal work it is wise to limit the types of apparatus. Familiarity with the material gives the child a sense of power and does not distract him from the number element in the examples he works.

In Appendix I a list of apparatus is given.

(4) **Faults in Organization and Method.** (*a*) *Too much Class-teaching.* Some children lose interest in arithmetic simply on account of the way in which they are taught. The 'hammer it in' method results in the brighter children becoming dull and bored, while the less gifted come to the conclusion that they are not meant to understand, and catch on to any facts or tricks which seem to succeed, and depend more and more on mere learning by rote. This is, of course, disastrous to their mental development. It is brought about as a rule by teachers talking and doing too much and forcing the children to follow without giving them sufficient opportunity for investigating and reasoning. As will be seen in the following sections, while individual work is accentuated, class-work and mental drill, both oral and written, are considered essential. The point is to find the proportion of each which succeeds best with the children concerned.

(*b*) *Faulty grading of the exercises* is another cause of failure. It is not an easy thing to grade arithmetic exercises. Many text-books for little children fail signally on this point. A study of the history of mathematics will prove illuminating, for while we cannot accept the 'culture epoch' theory in all its significance, we must allow that what has taken the race a long time to find out is likely to prove difficult to a child.

(In this connexion read *A Study of Mathematical Education*, by Benchara Branford.)

Fortunately, every teacher is not called upon to frame a syllabus. Excellent text-books are to be had in which the author has worked out and faced these difficulties for us —what remains for us to do is to choose these right books, and to add exercises to meet the needs of individual children.

(*c*) *Unsuitable Classification of Children.* Some teachers do not seem to be able to rest if their children are not all doing the same work. They speak about the "tail of the class," and unfortunately often use driving methods to shorten the length and improve the quality of this tail! There should never be a tail to a class; neither should there be children who constantly fail at tests. Every child should be following the course in which he is capable of doing good work.

Suggestions for carrying out such a scheme are given in the next section.

Section III. Practical Suggestions for carrying out an Arithmetic Scheme

The first essential is that there should be a carefully graded scheme for the school. This scheme should be continuous from the lowest to the highest class. The work and apparatus should be so arranged that there is always " a next thing " for each child to do, and each child should know what that next step is and where it is to be found. For children under eight a series of graded cards is best. A card appeals to a small child more than a book does—probably because it comes as something fresh; also, the work on it does not look so overpoweringly long. To work through a card with five sums on it is a very possible achievement. Hence cards on which many sums are crowded together lose their point—one might just as well use a book.

At definite points in the scheme there should be test-cards. These the teacher should keep and give out to individual children. For these little people no definite hour or day should be set aside for tests, neither should there be any excitement in connexion with them. They should just be taken as the " next thing to be done," and no extra importance should be attached to them. They are there to *help the teacher* to gauge the progress of each child.

The tests must be exactly on the work of the stage, so that any child who has conscientiously worked through the stage will pass the test. Several tests should be kept by the teacher for each stage, so that, should a child get below 60 per cent., he could be allowed, after practice, to try again with another test of the same difficulty.

Very little writing should be entailed in this early work. The cards should, for the most part, be filled in with digit-tablets. As the child gets older the amount of written work increases, until at about seven and a half or eight years of age he begins his first arithmetic text-book.

This is not the place to discuss the relative values of the innumerable text-books on the market. Suffice it to say

that, whichever is chosen, the preparatory course should be a direct preparation for it, and, secondly, the same book or series of books should be divided into stages, just as the preparatory work was. It is a good plan to put little coloured gummed stars in each book to mark the stages. When a child reaches a star he knows there is a test, and he should be allowed to do it at the next arithmetic period.

Some teachers feel that it cramps their teaching and lessens their interest when a text-book is prescribed and a definite course planned for the whole school. As a matter of fact, it should free their powers to concentrate on *method*.

There is an immense amount of original work possible even when the course is seemingly cut and dried as regards matter and sequence. To meet the difficulties of individual children, to arrange supplementary cards for them, to note points at which a collective class is needed, to prepare that class carefully, to diagnose all the mistakes made by the children, to think out series of mental drills—all these things require thought and care, and give scope for originality. Moreover, when one set of arithmetic books is decided on for the school it is only meant that this series shall form, as it were, the backbone of the course, and be a safeguard to its thoroughness and sequence.

When a course is thus carefully planned there is no abrupt change from infant to junior work. The first six stages would probably consist solely in exercise-cards and apparatus ; then would follow stages in which a text-book is used in conjunction with cards and apparatus ; then later still the text-book would provide most of the work, though it would be supplemented here and there by exercise-cards, and when any new process was introduced apparatus would be available. In such a scheme as this each teacher might be given charge of a certain portion of the syllabus. For instance, Teacher A follows up all children doing Stages 5, 6, and 7, and gives the necessary collective lessons to groups of children. Teacher B takes Stages 8, 9, and 10, and so on. In this way the children pass up from one teacher to another, irrespective of their classes. This arrangement, of course, involves all arithmetic lessons being given at the same hour.

Section IV. Stimuli to Work

Though some writers of to-day deny the fact that there are any lazy children, nevertheless all must agree that children do not always feel inclined to be active just in the way the teacher would like or on the subject most calculated to benefit them later! In spite of all a teacher may do to make the children interested in arithmetic, there will always be some who will be inclined to waste time and shirk difficulties. Moreover, learning tables and working drill-cards cannot always be interesting! We all know what it is to feel disinclined to take trouble, to face drudgery, to do the hard rather than the pleasant thing. Therefore, just because a child is a child and is being trained it is allowable and even necessary to help him to make acts of the will by some sort of recognition of effort and achievement. This is particularly true of children over eight years of age. To the younger ones all is new and interesting, and they have practically no drudgery to face, but at nine and ten years of age new difficulties arise; a definite amount of uninteresting work has to be done—tables must be memorized, etc. At this age a child has a healthy desire to excel, and it is legitimate to make use of this, provided precautions are taken to avoid discouraging the less gifted and so inhibiting their best efforts.

The following are some methods of helping children to work steadily:

(1) Individual record-keeping.
(2) School records.
(3) Tests at the end of each stage.
(4) Group or team competitions.
(5) Group or team records.
(6) Rewards.

Throughout we must bear in mind in using any of the above that at best they are artificial props—only useful temporarily in helping a child to form habits and to get a 'foothold' in a subject. They should be completely discarded before a child leaves school; hence they are most useful in the middle school, between the ages of nine and fifteen.

(1) **Individual Record-keeping.** There are many ways of keeping records of progress. Some teachers like the form in which the exact achievement is indicated (see Fig. 1)—*e.g.*, recognition of figures 1 to 6. This is convenient when the children are only just beginning number, but it becomes impossible further up the school. Where a scheme for the whole school has been made out as suggested in Section III record-keeping becomes a very simple matter. The following method is suggested.

Name of child	Recognition of figures 1-6	Composition of Nos. to 6	Composition of Nos. to 10	Subtraction of Nos. below 10	
K. Benson					
M. Brown					
F. Carter					
A. Clarke					

Fig. 1

A card about the size of a post-card is given to each child when he begins a new stage. The card is divided in such a way that there is a space in which a date or a cross may be put for every step of the stage which the child works (Fig. 2). At the bottom of the card there is a space for the group letter or mark obtained for the test on that stage. When a child has completed a stage he is given a new card, and the old one is filed by the teacher for future reference.

Name T. Robertson. Stage. 4			
Step 1		Step 7	
" 2		" 8	
" 3		" 9	
" 4		" 10	
" 5		" 11	
" 6		" 12	
Test			
Supplementary Cards			
Remarks			

Fig. 2. INDIVIDUAL RECORD-CARD

Name of child	Stage 1		Stage 2		Stage 3	
	Work	Test	Work	Test	Work	Test
A. Adams						
C. Atkins						
W. Bell						
G. Coates						

Fig. 3

(2) **School Records** (Fig. 3). Large record-cards should be kept for the whole school. When a child has completed a stage and gives up his individual card the date is entered in

the column provided for that stage. Some schools persist in the class system, and hence these large record-cards are constantly having to be renewed owing to children being promoted to higher classes. The school-charts of progress should be kept in the hall. As only stages are marked, filling them in is not a great strain on the individual teacher's time.

The head teacher can then see at a glance what progress each child is making, and can readjust and reorganize accordingly. Every teacher should have a copy of the syllabus; this should show clearly the exact work for each stage, and give supplementary exercises and the apparatus to be used in the stages for which the teacher is responsible.

(3) **Tests at the End of Each Stage.** The results of these are best classified as "Very good," "Good," or "Satisfactory"; or as A, B, C, D. *As the children should never attempt a test until they are ready for it, a mark lower than "Satisfactory" or C should very rarely be required.* In such a case the child should use supplementary exercises, and then work a second test of the same standard.

(4) **Group or Team Competitions.** When children are nine or ten years of age their desire to excel may be made use of to help them to master the mechanical part of their work. Some system of team- or group-work is best. In the 'bees' or other forms of competition it is essential that the matter be very definite, and well within the experience of the child. *Unless a child is ripe for the work of memorizing, it does more harm than good.* Hence this work should be carefully indicated in the stages of the school scheme, and nothing should be included in which the child has not had the necessary preparatory experience. For example, the table of sevens should be memorized only after the child has built it up and has used its combinations again and again in his exercises. Children need careful training in the best methods of memorizing, and they should be taught to recognize the psychological moment for making a final attack on some set of facts.

(5) **Group or Team Records.** For children over nine or ten who have been so well taught as not to have lost confidence

in their own powers the team or house system of marking may give excellent results. A child weak in arithmetic may win good marks for his side in English or history. There are, however, great dangers in any system of marking that is carried out elaborately.

(*a*) The children in order to get good marks may neglect their weak subjects or lose interest in them.

(*b*) The less gifted children will tend to learn facts by heart, and the more gifted, if not actually kept back, are held too rigidly to a set plan ; the result is often a dead level of achievement in a class consisting of children of very varying capabilities. This is, of course, more noticeable in subjects like literature and history than in arithmetic.

Where a system such as that described in Section III is used—in which the child works at his own rate and is marked according to his progress, whether he be at Stage 3 or at Stage 12—the weaker children are shielded by working at a lower stage, but there always remains the danger of their being too much hurried.

(6) **Rewards.** If it is considered advisable to give rewards these should be in the form of certificates for stages satisfactorily worked. On no account should work be rewarded by play. We grown-ups often preach to children the value and honour of work, and then we reward it by giving a tea-party or allowing a romp ! Work should be rewarded by special opportunities and facilities for further work. Instead of a holiday, why not give a free work day, when every child can make his own time-table and work at his favourite subject ? It will often be found that the children will organize themselves into groups and teach one another a great deal.

It remains to say a few words with regard to punishments and the publication of results.

Punishments inflicted for unsatisfactory work are always signs of weak discipline or poor teaching. The consciousness that a punishment is sure to follow unsatisfactory work or that a scathing remark will be made often ruins a child's chance of doing well.

If lists must be read out and weekly achievements be proclaimed, some such arrangement as the following will prevent the undesirable results previously mentioned :

"A. Brown has worked three supplementary exercises."
"E. Smith has passed Test 18."
"M. Jones has worked satisfactorily at Stage 5."

This would simply be an account of work done and not imply competition. The names of children who for any reason had not done good work would of course be omitted.

Section V. Individual Work

The following are a few short notes on how to organize individual work in a school:

(1) The children should have tables, not desks with sloping tops.

(2) The apparatus should be kept in low cases or cupboards easily accessible to the children.

(3) The apparatus should be carefully graded, and that suitable to each step should be easy to find.

(4) The children should fetch and put away their own apparatus.

(5) The teacher should go round and correct the children's work. She should guard against helping a child who really could get out of a difficulty himself.

(6) When a child has completed a piece of work that needs correcting he should be trained to go on to something else if the teacher cannot attend to him at once.

(7) Each child should have a box of his own in which to keep his pencil, long bead-bar, digit-tablets, record-card, etc., and some occupation which he may turn to in spare moments when tired or while waiting for the teacher.

(8) Talking should be allowed. If the class is working well this will never develop into more than a happy buzz.

(9) Children should be allowed to help one another.

(10) To assure herself of the reality of the progress of each child the teacher should watch each one working through an exercise from time to time. The elder children should be

tested regularly as suggested in Section III. Before passing on to a new stage every child should be questioned individually on the stage which has just been completed.

The following are some practices which should be carefully avoided:

(1) Giving out the apparatus to the children instead of letting them fetch it themselves.

(2) Limiting a child to one exercise and, when that is finished, insisting on his sitting still or doing some other subject.

(3) Breaking the time up into short periods. Individual work has proved that a small child can concentrate for very considerable periods. When he is interrupted every quarter of an hour he does not get a chance of resting or of concentrating. If the children fetch their own work and are allowed to move freely round the room, and even to *do nothing* for a time, then an hour on end is not too much to give to individual work.

(4) Allowing the children to crowd round the teacher's table waiting to have their work corrected. As a rule the teacher should go to the children, though it is allowable for her to take a group to her table for special help.

(5) Insisting on rigid silence, which can be maintained only at too high a price, and which at best is not conducive to happy work.

CHAPTER II

NUMBER IN THE NURSERY

SECTION I. THE AIMS OF THE TEACHER

CHILDREN who come to school at six years of age from well-to-do, cultivated homes are usually found to be far in advance of children from poorer homes who have been in school since the age of four. The home child has, as a rule, a better vocabulary, speaks more distinctly and fearlessly, and knows how to play. In fact, being accustomed to intercourse with grown-up people, he is more at home and natural with them, and takes a keener interest in all that is going on around him. If he is behind in reading and other formal school-work his background of experiences and power of expressing himself soon enable him to make good this deficiency.

The aim of a teacher in the baby-room ought to be to give the children the same experiences and opportunities for expressing themselves as children in good homes have in their nursery life. It is impossible to succeed wholly in this; for the number of children in the class, the short hours they are at school, and the home influence are only a few of the many difficulties in the way. But at any rate as far as number ideas are concerned the school-child ought not to be at a disadvantage. Children are very suggestible, and need only invitations to learn and scope for their activity. What is requisite is thought and planning on the teacher's part, and alertness to make the best use of opportunities for incidental teaching. The important point is not to give a lesson, for instance, on halves and quarters, but to introduce the idea of halving and quartering into the children's activities.

There are three chief ways in which the baby-room or nursery may prepare the child for later work in arithmetic:

(1) By providing opportunities for experiences which will familiarize him with the simpler number ideas and

with the activities on which number calculations are based (see Section II).

(2) By helping him to acquire the necessary vocabulary (see Section III).

(3) By training him to handle toys and apparatus gently (see Section IV).

Section II. Experiences which give Some of the Fundamental Ideas underlying Arithmetic

The activities in the nursery should give rise to ideas of length, area, volume, weight, quantity, shape, time, and the nature of different materials.

The following activities are recommended :

(1) Reciting, singing, and acting nursery rimes.
(2) Sorting and counting objects. Guessing games.
(3) Counting actions. Games of skill.
(4) Constructive play.
(5) Imitative and imaginative play.
(6) Use of special toys and individual apparatus.

(1) Whenever possible the children should act the rimes. Very helpful ideas for this will be found in *Spoken Poetry in the Schools*, by M. Gullan. In many cases, however, less formal acting should be encouraged. The rime may often form an introduction to some activity, as, for instance, " Smiling Girls and Rosy Boys," which would lead to children answering the call and coming to buy with cardboard coins.

In the following selection some of the rimes have been included simply for the sake of the help they give the child in acquiring a vocabulary (see next section).

Rimes giving the results of addition and multiplication tables have not been included, for they should not be taught until the child has built these tables up with concrete material and has *found the results for himself*.

(*The first two rimes are for counting on fingers or toes*)

This little pig went to market [*thumb or big toe*],
This little pig stayed at home,
This little pig had roast beef,
This little pig had none,
This little pig went " wee, wee, wee " all the way home.

Thumbkin says, " I'll dance," Thumbkin says, " I'll sing,"
[*Thumb moved backward and forward.*]
Dance and sing, ye merry little men, [*All fingers moved.*]
Thumbkin says, " I'll dance and sing." [*Thumb moved alone.*]

Pointer says, " I'll dance," Pointer says, " I'll sing,"
Dance and sing, ye merry little men,
Pointer says, " I'll dance and sing."

Long Man says, " I'll dance," Long Man says, " I'll sing,"
Dance and sing, ye merry little men,
Long Man says, " I'll dance and sing."

And so on with Ring Man and Little Man.

Two little blackbirds,
Sitting on a hill.
This one Jack,
This one Jill.
Fly away, Jack,
Fly away, Jill.
Come back, Jack,
Come back, Jill.

One, two, three, four, five,
Catching fishes all alive.
Why did you let them go ?
Because they bit my finger so.

One, two, three, four, five,
I caught a hare alive ;
Six, seven, eight, nine, ten,
I let her go again.

A long-tailed pig, or a short-tailed pig,
Or a pig without any tail,
A small pig or a big pig,
Or a pig with a curly tail.

One, two, three, four,
Mary at the cottage door ;
Five, six, seven, eight,
Eating cherries off a plate,
O-u-t spells out.

One, two, kittens that mew ;
Two, three, birds on a tree ;
Three, four, shells from the shore ;
Four, five, bees from the hive ;
Five, six, little hayricks ;
Six, seven, rooks in the heaven ;
Seven, eight, sheep at the gate.

Long Time Ago

Once there was little Kitty
Whiter than snow.
In the barn she used to frolic
Long time ago.

In the barn a little mousie
Ran to and fro,
And she spied the little Kitty
Long time ago.

Two black eyes had little Kitty,
Black as the sloe,
And they spied the little mousie
Long time ago.

Four soft paws had little Kitty,
Soft as the snow,
And they caught the little mousie,
Long time ago.

Nine pearl teeth had little Kitty
All in a row,
And they bit the little mousie
Long time ago.

When the teeth bit little mousie
Mousie cried out, "Oh!"
But she got away from Kitty,
Long time ago.

There were once two cats of Kilkenny,
Each thought there was one cat too many!
So they fought and they fit,
And they scratched and they bit,
Till excepting their nails
And the tips of their tails
Instead of two cats
There weren't any.

Solomon Grundy,
Born on a Monday,
Christened on Tuesday,
Married on Wednesday,
Very ill on Thursday,
Worse on Friday,
Died on Saturday,
Buried on Sunday,
This is the end
Of Solomon Grundy.

Christmas is coming, the geese are getting fat,
Please to put a penny in the old man's hat.
If you haven't got a penny, a halfpenny will do,
If you have not got a halfpenny, God bless you.

Black Friars

Seven black friars sitting back to back
Fished from the bridge for a pike or a jack.
The first caught a tiddler, the second caught a crab,
The third caught a winkle, the fourth caught a dab,
The fifth caught a tadpole, the sixth caught an eel,
And the seventh one caught an old cart-wheel.

One, two, buckle my shoe ;
Three, four, knock at the door ;
Five, six, pick up sticks ;
Seven, eight, lay them straight ;
Nine, ten, a good fat hen ;
Eleven, twelve, dig and delve ;
Thirteen, fourteen, maids are courting ;
Fifteen, sixteen, maids in the kitchen ;
Seventeen, eighteen, maids are waiting ;
Nineteen, twenty, my platter's empty.

Two-legs sat upon three-legs
With one-leg in his lap ;
In comes four-legs,
And runs away with one-leg ;
Up jumps two-legs,
Catches up three-legs,
Throws it after four-legs,
And makes him bring back one-leg.

There were three crows sat on a stone,
  Fal la la la lal de,
Two flew away and then there was one,
  Fal la la la lal de.
The other crow finding himself all alone,
  Fal la la la lal de,
He flew away and then there was none,
  Fal la la la lal de.

Buttons a farthing a pair,
Come, who will buy them of me ?
They are round and sound and pretty,
And fit for girls of the city.
Come, who will buy them of me ?
Buttons a farthing a pair.

A pye sat on a pear-tree,
A pye sat on a pear-tree,
A pye sat on a pear-tree,
Heigho! heigho! heigho!
And once so merrily hopped she,
And twice so merrily hopped she,
And thrice so merrily hopped she,
Heigho! heigho! heigho!

Pairs of Pears

Twelve pairs hanging high,
Twelve knights riding by,
Each knight took a pear,
And yet left a dozen there.

Twelve Huntsmen

Twelve huntsmen with horns and hounds,
Hunting over other men's grounds;
Eleven ships sailing o'er the main,
Some bound for France and some for Spain;
I wish them all safe home again;
Ten comets in the sky,
Some low and some high;
Nine peacocks in the air;
I wonder how they all came there?
I do not know and I do not care;
Eight joiners in Joiners' Hall,
Working with the tools and all;
Seven lobsters in a dish,
As fresh as any heart could wish;
Six beetles against the wall,
Close by an old woman's apple-stall;
Five puppies of our dog Ball,
Who daily for their breakfast call;
Four horses stuck in a bog;
Three monkeys tied to a clog;
Two pudding-ends would choke a dog,
With a gaping, wide-mouthed, waddling frog.

Thirty days hath September,
April, June, and November.
All the rest have thirty-one,
Excepting February alone,
Which hath but twenty-eight days clear,
And twenty-nine in each leap year.

The Twelve Days of Christmas

The first day of Christmas my true love sent to me
A partridge on a pear-tree.

The second day of Christmas my true love sent to me
Two turtle-doves and a partridge on a pear-tree.

The third day of Christmas my true love sent to me
Three French hens, two turtle-doves,
And a partridge on a pear-tree.

The fourth day of Christmas my true love sent to me
Four colly birds, three French hens,
Two turtle-doves, and a partridge on a pear-tree.

The fifth day of Christmas my true love sent to me
Five gold rings, four colly birds,
Three French hens, two turtle-doves,
And a partridge on a pear-tree.

The sixth day of Christmas my true love sent to me
Six geese a-laying, five gold rings,
Four colly birds, three French hens, two turtle-doves,
And a partridge on a pear-tree.

The seventh day of Christmas my true love sent to me
Seven swans a-swimming, six geese a-laying,
Five gold rings, four colly birds,
Three French hens, two turtle-doves,
And a partridge on a pear-tree.

The eighth day of Christmas my true love sent to me
Eight maids a-milking, seven swans a-swimming,
Six geese a-laying, five gold rings,
Four colly birds, three French hens, two turtle-doves,
And a partridge on a pear-tree.

The ninth day of Christmas my true love sent to me
Nine drummers drumming, eight maids a-milking,
Seven swans a-swimming, six geese a-laying,
Five gold rings, four colly birds, three French hens,
Two turtle-doves, and a partridge on a pear-tree.

The tenth day of Christmas my true love sent to me
Ten pipers piping, nine drummers drumming,
Eight maids a-milking, seven swans a-swimming,
Six geese a-laying, five gold rings, four colly birds,
Three French hens, two turtle-doves,
And a partridge on a pear-tree.

The eleventh day of Christmas my true love sent to me
Eleven ladies dancing, ten pipers piping,
Nine drummers drumming, eight maids a-milking,
Seven swans a-swimming, six geese a-laying, five gold rings,
Four colly birds, three French hens, two turtle-doves,
And a partridge on a pear-tree.

The twelfth day of Christmas my true love sent to me
Twelve lords a-leaping, eleven ladies dancing,
Ten pipers piping, nine drummers drumming,
Eight maids a-milking, seven swans a-swimming,
Six geese a-laying, five gold rings, four colly birds,
Three French hens, two turtle-doves,
And a partridge on a pear-tree.

Rosey Posey

Rosey Posey gets up at eight,
Goes to school and never is late ;

Rosey Posey dines at one,—
When her lessons and sums are done.

Rosey Posey at five has her tea,
Dolls and kittens invited free,

Rosey Posey plays at six,—
Builds a beautiful house of bricks,

Rosey Posey at seven o'clock
Takes off pinafore, shoe and sock ;

Eight by the clock she's tuck'd up cosy,—
End of day for Rosey Posey ![1]

Measuring

Measure the wool,
And measure the yarn,
And count the sheep in your father's barn.

Measure the thread,
And measure the silk,
And count the buckets of creamy milk.

Measure the barley,
And measure the grain,
And put your right foot out in the rain.

Measure the oats,
And measure the rye,
And put your left foot out to dry.

Measure the hummock,
And measure the hill,
And reach for the mountain higher still.

Measure the clouds,
And measure the sky,
And stretch your hands to the birds that fly.

[1] From *Pillow-land*, by Clifton Bingham (Boston Music Co.).

Measure the windmill,
To and fro,
And round and round at last you go ![1]

How many days does baby play ?
Saturday, Sunday, Monday,
Tuesday, Wednesday, Thursday, Friday,
Saturday, Sunday, Monday.[2]

"BANK"

What will you give me for my pound ?
Full twenty shillings round.
What will you give me for my shilling ?
Twelve pence to give I'm willing.
What will you give me for my penny ?
Four farthings, just so many.[2]

How many seconds in a minute ?
Sixty, and no more in it.
How many minutes in an hour ?
Sixty for sun and shower.
How many hours in a day ?
Twenty-four for work and play.
How many days in a week ?
Seven both to hear and speak.
How many weeks in a month ?
Four as the swift moon runn'th.
How many months in a year ?
Twelve the almanack makes clear.
How many years in an age ?
One hundred says the sage.
How many ages in time ?
No one knows the rhyme.[2]

(2) **Sorting and Counting Objects.** When a child can say the number series it does not mean that he can count. Miss Drummond shows this clearly in *The Dawn of Mind* :[3]

He may begin his counting too soon, or he may say two or more number names without setting aside the objects that correspond. Even after the notion of number is considerably developed, he often shows failure to grasp the exact nature of the counting process. Binet finds that while nearly all children can count four things correctly at five years of age, they do not count thirteen things accurately until they are six.

[1] From *Stories to tell the Littlest Ones*, by Sara Cone Bryant (Harrap).
[2] From *Sing-Song*, by Christina Rossetti (Macmillan).
[3] Published by Edward Arnold and Co.

The first notion of number may be that of position in a series rather than of total number in a group. If we are counting things to a little child he may startle us by the senseless query, "Which is three?" In such a case he may think that three is the name of the particular object pointed to, or he may think that three means a definite place in a series; that is, his notion of number may be ordinal rather than cardinal.

Any tendency to think that the number names are names of particular things is soon checked by their application to very many different objects. But it may take longer before the child realizes that each number word stands for the whole group of objects already counted, and does not apply only to that last reckoned.

In some cases small groups up to three or four are recognized before the child is really able to count, and with these groups little arithmetical operations are often spontaneously performed.[1]

The following are a few suggestions for exercises:

Sorting and counting treasures found on nature-study rambles—shells, cones, pebbles, burrs, etc.

Collecting bus- or tram-tickets and stamps, and tying them up into packets of five or ten. Sorting them according to value.

Verifying number of bricks, counters, etc., in each child's box.

Counting out cups, saucers, and plates for dolls' tea-parties, etc.

Arranging Noah's Ark animals in twos, or soldiers in lines of five or ten, etc.

Sorting nests of boxes or trays, and fitting them into one another.

Sorting boxes of different kinds of objects—*e.g.*, shells, buttons, reels.

Sorting "Tiny Tots" toys and arranging them in groups —*e.g.*, one bench, two houses, three trees, and so on.

Sorting tablets of different shapes—*e.g.*, rounds, squares, triangles.

Sorting tablets or bricks of same shapes, but of different sizes.

Arranging rods according to their lengths.

Sorting oblongs and squares, and arranging each set according to size.

[1] In this connexion see the chapter on mathematics in *The Psychology of the Common Branches*, by Freeman (Harrap).

Sorting heavy and light objects.
Threading beads in twos or threes.
Threading beads according to arrangement shown on a picture-card.
Playing with dominoes and placing them in a series.
In stick-laying to count sticks required to make chair, table, and other common objects.
Counting up and down line at drill.
' Falling in ' to lines or circles of three or four.
Dividing into sides.

Guessing Games. These should be very simple, and individual children should be encouraged to make them up and to ask the class for the answer. This induces the children to count the objects around them.

The following are examples of possible games:

" I am thinking of something that has six sides and eight corners."
" I see something that is round."
" I see five things all of the same size (or shape)."
" There is something in this room that measures exactly one yard." (Papers one yard long must be provided.)

Terms involving comparison, such as " larger," " shorter," " smaller," should be introduced—*e.g.*, " I see an oblong smaller than the top of this table."

A round game much appreciated by children is the following:

First child. I have one nose.
Second child. A bird has two legs.
Third child. A clover leaf has three leaflets.
Fourth child. A table has four legs.

And so on.

If a child cannot find anything for the next number he may take the same number as the child before him. Sometimes the children like to *exhaust* one number before proceeding to the next. Of course, the children should be allowed to take objects in the room at the moment; for instance, there might be seven flowers in a vase.

(3) **Counting Actions or Events.** This is much more difficult for a small child than counting objects. It is best introduced by activities in which each repeated action is of the same value. Some examples are:

Counting while skipping, bouncing a ball, etc.
Counting steps when walking, etc.—*e.g.*, three steps, two hops round the room.
Counting the beats of a metronome.
Counting the beats on a drum, claps of hands, and strokes of a bell.
Counting the number of times the child succeeds in throwing a ball into a basket, or in rolling one through a hoop, or in knocking down ninepins.

(Under games of this kind we might put high and long jumping. This entails estimating the distance jumped—as high, higher, etc. Also we may add games that entail running half-way, quarter-way, etc., down the room.)

Games of Skill. Games which entail competition, and the more complicated scoring that arises from the actions *not* being all of equal value, are not for the nursery. Little children enjoy the activity for its own sake, and do not trouble about its results. When forced to compete it is often clear that they do not know when they have won or lost! Hence ninepins, quoits, and ring and bridge games in which the pins, disks, pegs, and arches are numbered are not suitable.

At a later stage the children will enjoy them, provided the scoring be done by means of counters or a simple scoring-bar.

When counters are used each side has a saucer into which the child puts a counter every time he scores one mark.

A scoring-bar is a narrow piece of wood about four or five feet long, with holes bored along it at intervals of one inch. Coloured wooden pegs representing the sides are then moved along it according to the score made. If a red mark is put across the bar at every multiple of ten the children learn a great deal incidentally while marking.

(4) **Constructive Play.** The child's natural desire to construct will provide the teacher with many opportunities for incidental teaching.

The instinct of construction is very prominent in most children. Long before they come to school, children show an impulse to make things, though the things be only mud pies, or toy houses and bridges. This instinct should be further employed in schools. The various forms of handwork give it exercise; in them *constructiveness* is the instinct most active; and by incessant hammering and sawing, and dressing and undressing dolls, putting things together and taking them apart, the child not only trains the muscles to co-ordinate action, but accumulates a store of physical conceptions which are the basis of his knowledge of the material world through life.

Closely connected with, and perhaps not to be definitely distinguished from, the instinct of construction, is the instinct of manipulating objects, which is also difficult to distinguish from some of the more general innate tendencies, especially from the tendency to play and the still more general tendency to bodily activity. It is the instinct of manipulation which leads a child to take things to pieces, and often to commit what appear to be wanton acts of destruction. There is no doubt that the instinct of curiosity often co-operates with it.[1]

The forms of constructive play are so varied that it is possible to mention only some of the most useful materials:

Stone and wooden building bricks of all sizes.

Floor-blocks and short, light planks.

Wooden boxes for making dolls'-houses, stables, etc.

Tiles of various sizes and shapes.

Cards for building houses.

Picture-cubes for building towers.

Sand, with spades and pails and moulds for making shapes and shells for making patterns.

Clay or plasticine.

Noah's Ark and farmyard animals, for which shelters may be built.

Small wheelbarrow, truck, or large cart and horse which can actually be used for transporting things.

Wooden cases containing material such as cotton-reels, mantle-boxes, match-boxes, pieces of wood and cardboard, newspaper, and wallpaper.

(5) **Imitative and Imaginative Play.** The children's constructive work very often centres round some of their favourite

[1] Dumville, *Child Mind* (University Tutorial Press), p. 91.

make-believe play. It will be noticed that if children of six and seven are left to themselves one game reigns supreme for a time. In a period during which 'tram' games are the fashion a child will give away all his toys for the sake of acquiring tram-tickets or other objects immediately connected with the absorbing interest of the moment. Though this intense form of make-believe is not, as a rule, found in the nursery stage, yet we see the beginnings of it there, and hence it seems to require some consideration here.

Montessorians hold that make-believe play should be discouraged as leading from the truth and cultivating unreality. It is, Dr Montessori asserts, the expression of "unsatisfied desire." "A form of imagination supposed to be 'proper' to childhood, and almost universally recognized as creative imagination, is that spontaneous work of the infant mind by which children attribute desirable characteristics to objects which do not possess them."[1] She then goes on to say "We, however, suppose that we are developing the imagination of children by making them accept fantastic things as realities." The children are credulous, they believe; we imagine and impose our imaginings on them, thus increasing their difficulty in distinguishing truth from falsehood. Froebelians, on the other hand, purposely introduce make-believe into the activities of their children, believing it to be an essential of play. But few of them would tolerate the exaggerated form found in some books by kindergartners. For instance, one author advocates that, when a child is using Tillich bricks, it should be suggested to him to call them "Mr Ten, Mrs Nine, Baby One," etc. Another, in a lesson in reading, leads the children to suppose that the letters are little boys and girls! *a* and *t* are two little boys; *a* goes into the playground and *t* follows him, and they hold hands thus, *at*. *c*, a little girl, comes out by another door and chases and catches them, so that we get *cat*! One can only say that such devices are unworthy of the subject-matter, and still more of the child. They are truly silly, shallow, and frivolous, and show a great lack of balance and proportion in the teacher. On the other hand, the Montessorians seem not to recognize the very real

[1] *The Advanced Montessori Method* (Heinemann), vol. i, p. 256.

knowledge and power that a child may acquire by spontaneous make-believe play, and how by degrees the experiences these plays furnish actually help him to an understanding and appreciation of truth. Make-believe is a natural phase in a child's development, and we do not help him when we try to suppress it or cut it out. A wiser plan seems to be to allow spontaneous, natural make-believe play, and tactfully to make use of the opportunities it affords for real teaching, without, however, imposing our imaginings on the children. Such games have many of the advantages usually attributed to dramatic work. The more usual games are:

"Houses"—which includes tea-parties, doctor's visits, and other events of family life.
"Travelling Games"[1]—such as "Trams," "Buses," "Trains," "Boats," "Coaches."
"Schools."
"Shops."
"Farmyard" and "Circus."

In addition to the materials and toys mentioned under "Constructive Play" the following are recommended:

A small, low seesaw—easily taken to pieces—on which children may balance themselves, as well as other objects.
A large, strong pair of scales, with weights.
Cardboard coins and tickets of all sorts.
Wooden rods measuring one yard and one foot.
Rolls of paper or braid for measuring out.
Bags of strong calico.
Measuring vessels (pint, quart, gallon, bushel, etc.).
Tea- and dinner-sets of a sensible size.
Boxes and bags containing material for measuring out and for stocking a shop; *e.g.*, maize, beans, pebbles, bran, sawdust, wool, paper torn up, etc.

A readymade miniature shop is not nearly so educative as floor-blocks and a light plank with which a counter can be built. A child can then sit on a stool and be 'shopman,' and weigh out ounces and pounds with real scales.

[1] Suggestions for these games will be found in *Education by Life*, edited by Henrietta Brown Smith (published by Philip).

A miniature shop, however, makes a very good subject for co-operative constructive work, but it is the *process of making* that is of value, not the finished article.

(6) **Use of Special Toys and Individual Apparatus.** This subject is very ably dealt with in the following passage:

> The phenomenon to be expected from the little child when he is placed in an environment favourable to his spiritual growth is this—that suddenly the child will fix his attention upon an object, will use it for the purpose for which it was constructed, and will continue to repeat the same exercise indefinitely. One will repeat an exercise twenty times, another forty times, and yet another two hundred times, but this is the first phenomenon to be expected, as initiatory to those acts with which spiritual growth is bound up. That which moves the child to this manifestation of activity is evidently a primitive internal impulse, almost a vague sense of spiritual hunger; and it is the impulse to satisfy this hunger which then actually directs the consciousness of the child to the determined object and leads it gradually to a primordial, but complex and repeated exercise of the intelligence in comparing, judging, deciding upon an act, and correcting an error. When the child, occupied with the solid insets, places and displaces the ten little cylinders in their respective places thirty or forty times consecutively; and having made a mistake, sets himself a problem and solves it, he becomes more and more interested, and tries the experiment again and again; he prolongs a complex exercise of his psychical activities which makes way for an internal development.[1]

There is no doubt at all that the Montessori sensory apparatus meets this " internal impulse " of the child. Each piece seems to give an outlet to a pent-up power of concentration and attention of which children are often considered incapable. Though the direct aim of the cylinders, insets, etc., is not to give knowledge, but to make the child " exercise his activities," yet the use of this apparatus does lead to definite and ordered knowledge.

For reasons set forth in the above quotation and in Section IV of this chapter, it seems that no baby-room is complete without some of Dr Montessori's apparatus. The cylinders, broad and long stairs, tower, and wooden and metal insets are valuable as having a bearing on arithmetic. They are recommended not because they are the only objects that

[1] Montessori, *The Advanced Montessori Method* (Heinemann), vol. i, p. 153.

meet a child's "internal impulse," nor because it is doubted that less formal apparatus could supply the need. The reason is simply that they are the result of much thought and research work, and are based on sound psychological principles. Moreover, they are ready to hand, accurate in every detail, durable, and, in some cases, beautiful. Other objects might be found to fix a child's attention ; for instance, a little boy was once seen fitting wooden boxes into one another, and the process was repeated again and again. Fitting and sorting exercises can easily be provided, but boxes have no knobs, and most of us know what a fascination a knob has for a child ! Moreover, having Montessori apparatus should not bind us to make use of no other—on the contrary, its presence should emphasize the necessity of providing beautiful toys, and by its dignity and reserve it should raise our standard of what is worthy of a child. A large Noah's Ark with animals all in proportion, a farmyard with fences, houses, animals, and utensils, can now be bought at very reasonable prices. These and other objects that are worthy of respect by reason of the perfection of their finish, their beauty, their fragility, or the delicacy or importance of their functions (*e.g.*, a weighing machine) should be specially cared for by the children.

SECTION III. VOCABULARY

A child must acquire a very definite vocabulary if he is to understand even simple problems.

The following are some necessary words : high, low, broad, long, short, narrow, deep, many, few, most, much, heavy, light, weigh, measure, ounce, pound, add, take away, share, divide, equal, remain, left over, greater, less, altogether, buy, sell, change, owe, whole, half, quarter, square, oblong, circle, triangle, hour, minute, second, names of cardinal and ordinal numbers, names of common coins.

Now to make any of these words part of his working vocabulary the child must learn :

(1) To say the word.

(2) To understand it when used by other people.

(3) To use the word himself in different connexions.

(4) To interpret it in action or drawing, and to recognize its representation in pictures, etc.

These four activities do not, of course, follow one another in the order given above. A child may understand a word before he articulates it, or be able to interpret it before he uses it himself or, indeed, before he thoroughly understands it. His interpretation may be simply imitation, or it may be limited to only one aspect or use of the word.

There are two processes by which we learn the meaning of words:

(1) We pick up the word, and by degrees discover its meaning and application, or

(2) We become familiar with an activity or object, and then the word comes as a summary of it.

(1) Some teachers seem too afraid of letting children use words they do not understand. While formal, parrot-like learning has rightly been condemned, we must not forget that children are naturally interested in sound as sound, and will repeat words that have attracted their attention simply for joy in their sound and with no desire of understanding them. After all, it is *something* to be able to say the word, and it is more likely that they will *find out* the meaning of a word they can say than of one they have heard, but have never articulated.

To have picked up a word and repeated it, and then gradually, through using it and hearing it used, to have realized its full meaning, is a very common way of learning, and one that goes on all through life. There is such a thing as over-planning a child's progress through a desire that he should learn a thing 'just so' and 'no otherwise.' The 'just so' method in the case of acquiring a vocabulary is discussed under (2). It is the scientific method, and at the same time the *natural way in which language first arose*, but it is inevitable that children should reverse the process, and it is fortunate that they do, otherwise their progress would be slow.

In school children often sing and recite rimes of which they understand the general meaning, but which contain words

that are meaningless to them out of their context. These words the teacher should introduce into their activities, and so by degrees teach their meaning.

On account of this fear of children using or hearing words they do not understand, some teachers in infants' schools not only 'talk down' to the children, but exaggerate and distort the articulation of words and assume a most unnatural pitch and tone! Sometimes it seems as if all the children must be deaf or imbecile! This is a very bad fault in a teacher, as it tires the children and hinders their development. We have only to talk to an averagely intelligent child brought up in a cultivated home to realize what a number of words a child can use, and what fluency of speech he can acquire if only his environment is stimulating.

(2) In planned work for the extension of a child's vocabulary the teacher introduces some activity—for instance, sorting rods of different lengths—and then waits until the child has become perfectly *familiar with the rods and their qualities.* Then and *then only* does she teach the words that name those qualities, 'long' and 'short.' The activity comes first, and the word afterwards as a kind of seal which defines and fixes the idea in the mind. From this it will be seen that a lesson to teach such words as 'long' and 'short' will directly attain its end only if the child is perfectly familiar with the qualities in objects he has handled; otherwise the words cannot come as a seal on the idea, nor can the child feel a need for a word to express, for example, longness and shortness; hence he will be left with only a vague idea of the qualities and perhaps with the power of pronouncing the words. In fact, he will have to *find out* the exact meaning of the words by degrees, as described in (1).

A vocabulary grows, and it cannot be added to piecemeal —each word must be bound up with the child's activities. The work should be very definitely arranged in the teacher's mind, but not in half-hour achievements! She should have a list of words she intends the children to acquire, and so suggest and guide their activities that each child has opportunity for learning them. We cannot save the child from all vague ideas and confused thoughts, but we *can give*

him a stock of very definite ones that will help him in his difficulties.

Section IV. Handling Toys and Apparatus

At first sight one might wonder what this had to do with arithmetic. But on reflection it becomes clear that unless a child learns to handle apparatus suitably his later practical work will be apt to lack precision and accuracy. His early training should prepare him to appreciate good instruments, and to dislike the very sight of chipped and inky rulers, blunt compasses, and bent protractors! All these things tend to encourage slipshod and inaccurate work.

Moreover, a child's touch and kinæsthetic sensations have so important a bearing on his ideas of space, size, and direction that they might well have been discussed under Section II —"Experiences which give Some of the Fundamental Ideas underlying Arithmetic." Indeed, what is said below under heading (3), "The Number and Variety of the Children's Touch and Muscular Experiences," applies to many of the exercises in Section II.

We may say that the way children handle apparatus will depend upon:

(1) The example given by the teacher.
(2) The children's understanding of the use of the objects.
(3) The number and variety of the children's touch and muscular experiences.
(4) The nature and condition of the apparatus.

(1) **The Example of the Teacher.** A teacher who shows little respect for books by throwing them down, opening them roughly, putting her finger or thumb to her lips before turning the pages, etc., must expect the children to follow suit. Indeed, not only will they imitate her definite actions, but the atmosphere of the room will tell upon their treatment of the things in it. Loud speaking, noise, untidy cupboards or drawers—in fact, disorder of any kind—will induce rough handling of apparatus. On the other hand, we have only to look into a well-ordered Montessori room to see how children respond to a carefully prepared environment.

(2) **The Children's Understanding of the Use of the Objects.** A child's attitude towards bean-bags, bricks, clay, reels, and other material provided for constructive work should be different from his attitude towards his pair of scales or his jigsaw puzzles. The little boy who filled the holes for the Montessori cylinders with water either did not understand the use of the apparatus or he had outgrown it. His attitude was wrong; he had not learnt the difference between constructive material, toys to make and experiment with, and objects that have been made expressly to serve a certain purpose, and to be used in a very definite way. It may be said that we limit the child's opportunities of adapting, inventing, and experimenting if we set some things apart for specific uses, but this limiting is a necessary part of education in the infant school. Respect for apparatus is not difficult to teach. All that has to be done is to explain the uses of the different things and to show the children how to dust and keep them clean. If children learn to take a pride in their apparatus, and if plenty of constructive material of the right kind is provided, there will be little danger of things being misused. In the many stories we hear of children's so-called 'mischief' almost always the things misused have not been understood and appreciated, or the child perpetrating the 'mischief' has not had opportunity for activity suited to his age and requirements.

(3) **The Number and Variety of the Children's Touch and Muscular Experiences.** The clumsiness and roughness of children come from their lack of muscular control, the craving they have to be active, and their ignorance of space-relations and of the properties of the various materials. Hence the more experience they have of touching, stroking, patting, lifting, and handling things, the more likely they are to become gentle, because they will realize the nature of the different materials and treat them accordingly. It is the 'feel' of satin, plush, or fur that invites a child to stroke them gently, and to be allowed to stroke trains the child in gentle ways. It is the lack of training of the sense of touch in the nursery that is at the back of much of the roughness we see in school-children. Gentleness in handling things is acquired only gradually, and

if early training has been lacking it is very difficult to put this right in the upper school.

"Every child needs a rich range of touch experiences—of the delicate for the appreciation of things refined, of the grosser for the appreciation of things strong, stately and sublime." [1]

Gesell goes on to say that the deep values that lie hidden in touch are "often vague and nearly always inarticulate." Hence immediate and tangible results are not to be expected.

The important connexion between a child's movements and his ideas of space, form, etc., are shown in the following extracts. Students are also referred to Rusk's *Introduction to Experimental Education* and to Stout's *Manual of Psychology*.

> When a child reacts to a given sensation, his reaction usually involves some form of movement, and the movement gives rise to kinæsthetic sensations, and the memory of these sensations, in the form of images, acts as a guide to his reaction on a future occasion. Thus he is able to profit by experience. . . . it is interesting to notice that this group of kinæsthetic sensations, which is not included in the usual list of the five senses, is nevertheless one of the most important, if not *the* most important group of sensations for the educative process of adaptation.[2]

And again:

> With regard to space the main categories are size, direction, distance, form, dimension (2 or 3), and the representation of tri-dimensional space in two dimensions as in a picture. Our perception of space is based in the last resort on our movements through space. Thus a young child learns his first lessons with regard to space by his movements of grasping or reaching towards an object and either touching it and finding that it is near him, or not touching it and finding it far away. In the second place, when he is old enough to creep or walk, he begins to learn about space by actually moving in it. In this way a gradually increasing and more accurate perception of size is built up. In order to improve the perception of form the elementary geometrical forms must be learnt and here the psychologically simplest is the circle or rather the sphere and not the triangle, as has often been suggested. Here again the surest way to attain to accurate perception is not by mere visual presentation, but by actual handling, as in the case of the Montessori sandpaper letters or the geometrical insets.[2]

[1] A. L. and B. C. Gesell, *The Normal Child and Primary Education.*
[2] Kennedy Fraser, *The Psychology of Education* (Methuen).

(4) **The Nature and Condition of the Apparatus.** The objects provided for the children should be worthy of their respect and care. It is not a question of new toys, but of suitable ones. It is a pity to provide only unbreakable things, as their constant use is apt to induce roughness in the children. There should of course be toys that will stand being hugged and thrown about, but there should be other things, perhaps not so lovable and comforting, but nevertheless much to be valued as being precious, and set apart for very special play. These " beautiful stand-off things," as a little girl called them, must be carefully looked after, dusted, and put away in a definite place. Any misuse of them will be quickly noticed by the children themselves, and will be rectified without the teacher having to interfere. As far as possible, these objects should be attractively coloured, and they should always be suitable for the age of the children who are to use them. When children are either too young to understand or too old to appreciate the use of a toy or piece of apparatus they are apt to misuse it.

CHAPTER III

NOTATION

I. RECOGNITION OF FIGURES. THEIR NAMES AND SYMBOLS

THE children will already have learnt the names of the numbers one to ten. In fact, the exercises described in the first three divisions of Section I will have been mastered in the baby-room or nursery. They are added here only for completeness' sake.

SECTION I. COUNTING AND THEN ASSIGNING A NAME TO NUMBER-RODS OR GROUPS OF DISCRETE THINGS

(1) **Counting Discrete Objects and assigning a Name to Groups of Them.** The formal exercises will consist in the teacher asking the child:

(*a*) To show three, four, etc., objects.

(*b*) To count how many objects she gives to him.

(*c*) To add objects to the number given so as to make another number; *e.g.*, the teacher gives the child two counters, and the child adds two to make four, counting out as he does so.

(2) **Assigning a Name to Groups of Pips arranged in a Definite Way.** Montessori recommends the vertical grouping in twos

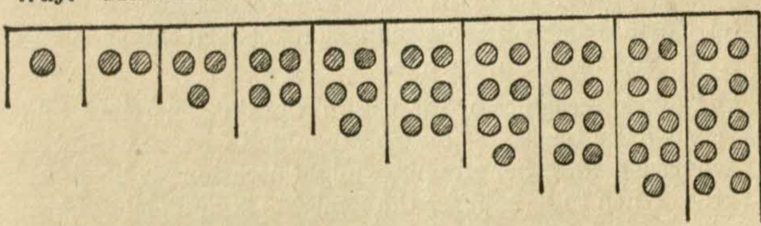


Fig. 4. GROUPING RECOMMENDED BY DR MONTESSORI

shown in Fig. 4. This has the great advantage of showing sequence, and odd and even numbers. Also every group is

readily converted by addition or subtraction into other groups. For quick recognition of isolated groups it is not so satisfactory, especially when arranged vertically, for our eyes are so constructed that they naturally move in a horizontal direction.

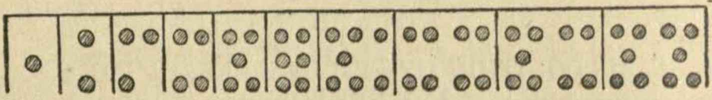


Fig. 5. Another suggested Grouping

McLellan and Dewey, in *The Psychology of Number*, recommend the grouping as in Fig. 5 :

> Real meaning is given to the operation of counting when, instead of using unarranged units, we have the rhythmic arrangement. The actual values of the measuring units, and the meaning of counting—necessarily related processes—are fully brought out. Six is at last perceived as six without the necessity of counting.

In the " Welbent " Series the use of the grouping to the basis of five (see Figs. 6 and 15) is advocated for the following reasons :

(*a*) It is excellent for quick recognition. The children easily visualize the square four and the compact five, and can recognize them even when isolated.

(*b*) It shows sequence clearly. Each number is built up from the one below it by the simple addition of one pip, without alteration in the positions of the others.

(*c*) It emphasizes the relation of each number to ten.

(*d*) Addition and subtraction of fives, the table of fives, and division by five are made very clear.

(*e*) There is an interesting connexion between this grouping and Roman numerals.

For quick recognition, for use in working exercises, and for wall-charts the same grouping should always be used, but on exercise-cards other groupings might occasionally be given. There is much to be said for the straight six and eight shown on dominoes. They certainly bring out the connexion of 2 and 3 to 6, and of 2 and 4 to 8.

The grouping in threes has its upholders (see Fig. 8). These maintain that three is the largest number of pips a child

can see at once. Moreover, they hold that multiplication and division by three are difficult processes, and that this grouping prepares a child for them. There is, however, a

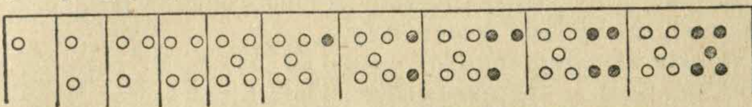


Fig. 6. Grouping used in the "Welbent" Series

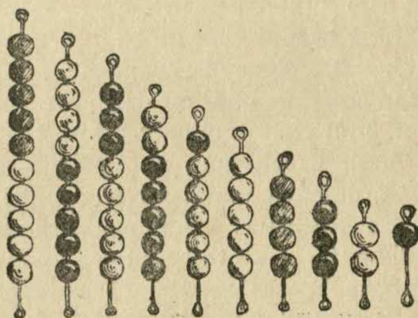


Fig. 7. "Welbent" Bead-bars

great lack of individuality in the groups, which makes them difficult to recognize quickly. They have not the advantage of the Montessori grouping, for they do not show odd and even numbers clearly—in fact, the 4 and 10 look like odd

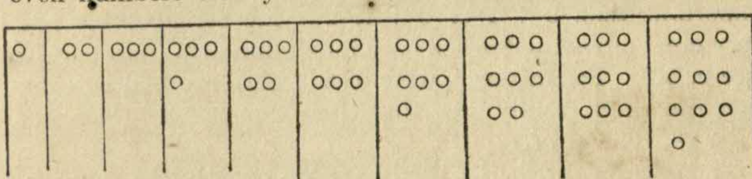


Fig. 8. Grouping in Threes

numbers, and the 9 like an even number. Moreover, to look on ten as 3 + 3 + 3 + 1 is not natural. This grouping is therefore not recommended.

The exercises in assigning names to groups of pips might be on the same lines as those described in (1) above.

(3) **Assigning a Name to each of the Short Bead-bars.** There are various kinds of bead-bars on the market.

(*a*) The Montessori "short stair" bead-bars consist of five

bars of each of the numbers 1 to 10—*i.e.*, fifty bars in all. Each number has its characteristic colour—*e.g.*, all tens are golden. The beads cannot be moved up and down the bar; hence they are not easy to count. A child comes to recognize the numbers by their colours only.

(*b*) The Chelsea bead-bars consist of (1) closed bars each with ten beads of the same colour. Unfortunately, too many colours are used for these 10-bars, so that there are red, yellow, green, and blue bars. (2) Open wire bars on to which loose beads may be strung to represent units. When there are ten beads on a bar it is full, and may be changed for a closed 10-bar.

The chief disadvantages of these bars are that the loose beads roll about, and that numbers below ten have to be built up from ones, which does not encourage adding in groups. The first difficulty may be overcome by using cubic beads, but they are expensive. Here again the length of the wires does not allow the beads to be moved up and down.

(*c*) The "Welbent" bead-bars (Fig. 7) are arranged to the basis of five to correspond with the grouping in Fig. 6. The wires are sufficiently long to allow a child to separate the beads into groups and to slide them along as he counts. Each bar above 5 has beads of two colours. Thus seven is recognized not by colour only, but also by its composition —viz., 5 + 2. This combination is, of course, no more important than 3 + 4 or 6 + 1, but the arrangement has very great advantages when used for all numbers from 1 to 10 (see p. 50). For instance, when two 7-bars are laid side by side the child readily sees that their sum is 14. (See also Figs. 26, 76, and 77.)

The exercises with the bead-bars are exactly the same as those with the number-rods described in the next section.

(4) **Counting the Divisions on the Numerical Rods[1] and assigning a Name to each particular Rod.** The Montessori numerical rods have divisions of one decimetre, so that the longest rod is one metre. This is inconveniently long for the ordinary class-room table or desk, though the arm exercise entailed when a small child lays them out is beneficial.

[1] These are on the same principle as Tillich's bricks used in the eighteenth century.

That some rods be used is desirable, as they give the child an opportunity of comparing and measuring wholes of different magnitudes, and lead him to realize that any number may be used as a unit with which to measure. The 1-inch number-rods are recommended as clear, strong, light, and of convenient size for class-room use.

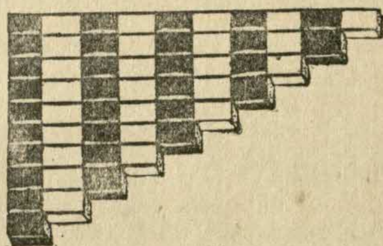


Fig. 9. 1-inch Number-rods

(*a*) The child counts the divisions on each rod and names them the 1-rod, 2-rod, 3-rod, 4-rod, etc., and lays them out as shown in the diagram (Fig. 9).

(*b*) When asked for any rod the child counts, and so finds it.

(*c*) He is then required to find the rod next in length to the one presented to him by the teacher.

(*d*) When the symbols or figures are introduced the child may put a figure-card on each rod.

(5) **Counting up and down the 10-reel Bar** (Fig. 10). Here the child simply moves the reels along as he counts backwards

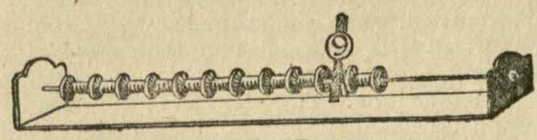


Fig. 10. The Reel-bar

and forwards. He can also be asked to show a definite number or to name a number marked off for him.

Section II. Recognizing the Symbols that represent the Numbers

(1) **Montessori Method.** In these exercises the visual, muscular, and tactile sensations are used to impress the shape or form of the figure on the child's mind. Mounted Montessori sandpaper or " Welbent " embossed paper figures should be used. (See Fig. 11.)

Simple Lessons on the Séguin Plan. (*a*) The teacher gives

the child a figure saying, "This is two," or "This is one," as the case may be.

The following quotation from *The Montessori Method* (p. 276),[1] though referring to the letters of the alphabet and their sounds, will be found useful here:

> As soon as we have given the sound of a letter, we have the child trace it, taking care to show him how to trace it, and if necessary guiding the index finger of his right hand over the sandpaper letter in the sense of writing.
>
> 'Knowing how to trace' will consist in knowing the direction in which a given graphic sign must be followed.
>
> The child learns quickly, and his finger, already expert in the tactile exercises, is led, by the slight roughness of the fine sandpaper, over the exact track of the letter. He may then repeat indefinitely the movements necessary to produce the letters of the alphabet, without the fear of the mistakes of which a child writing with a pencil for the first time is so conscious. If he deviates, the smoothness of the card immediately warns him of his error. The children, as soon as they have become at all expert in this tracing of the letters, take great pleasure in repeating it with closed eyes, letting the sandpaper lead them in following the form which they do not see. Thus the perception will be established by the direct muscular-tactile sensation of the letter. In other words, it is no longer the visual image of the letter, but the tactile sensation, which guides the hand of the child in these movements, which thus become fixed in the muscular memory.
>
> There develop, contemporaneously, three sensations when the directress shows the letter to the child and has him trace it: the visual sensation, the tactile sensation, and the muscular sensation.
>
> In this way the image of the graphic sign is fixed in a much shorter space of time than when it was, according to ordinary methods, acquired only through the visual image. It will be found that the muscular memory is in the young child the most tenacious and, at the same time, the most ready. Indeed, he sometimes recognizes the letters by touching them, when he cannot do so by looking at them. These images are, besides all this, contemporaneously associated with the alphabetical sound.

The teacher then gives a second figure, proceeding in exactly the same way as with the first. Two figures are as a rule enough for a child to learn in one lesson.

(*b*) The child has now two figures, and is given time to compare them. The teacher then asks for one of them,

[1] Published by Heinemann.

saying, "Give me two." If the child cannot recognize the figure by sight he should be encouraged to trace it again.

(c) In this step the teacher takes up one of the figures and asks, "What is this ?" This is more difficult, for it requires the child to say the word.

Important features of this method are :

(a) The simplicity and fewness of the teacher's words.

(b) The very definite response that is required from the child.

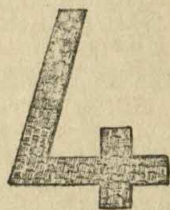


Fig. 11. "WELBENT" EMBOSSED FIGURE

It is by these means only that clearness and accuracy will be arrived at with very small children.

The child should be taught to pass his finger over the sandpaper or embossed figure with a very light touch. Undue pressure will distract him from the form of the figure he is tracing, so also will too great irritation of the nerves of the finger-tip. For this reason those children who show the slightest dislike to the 'feel' of sandpaper should use the embossed figures.

It is most essential that the child concentrate, not on the feel of the surface of the paper, which is meant only to guide his finger, but on the way he moves and the shape he traces out.

It does not really signify in what order the figures are taught. Some teachers prefer to teach the figures 1 to 6 first, and then to allow the children to add and subtract with these before proceeding to 10. One thing is certain, and that is that we should not stop at 9 just because the symbol for 10 is composed of two figures. The 10 should be presented to the child without comment. This is not slurring over a difficulty, but merely shows patience in waiting until the child is ready to understand the explanation. (See Chapter IV.)

The counting exercises detailed in the following chapters are calculated to prepare the child for the understanding of place-value, and by their use the teacher will be able to detect the best moment for giving an explanation, which should be followed immediately by formal exercises on notation.

(2) **Counting out and placing Objects in Groups beside the Corresponding Figures.** (*a*) A figure or digit-tablet is put into each of the five compartments of a tray (see Fig. 12).

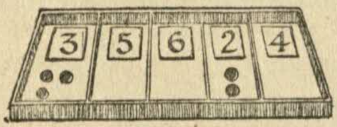


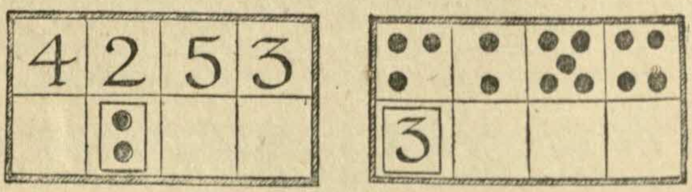
Fig. 12. Test-tray

These compartments are large enough to allow of the child arranging his counters in distinctive groups. Montessori test-boxes serve this purpose, but, besides being expensive, the spindle-shaped sticks, when the number exceeds 4, are apt to lie on one another in such a way that no good result for visualization is obtained.

(*b*) In the "Welbent" Series two kinds of cards are used:

(i) With figures given and spaces left below each for the child to fill in with the corresponding group. (See Fig. 13.)

(ii) With groups of pips given and space below each for the child to fill in with the corresponding figure. (See Fig. 14.)



Figs. 13 and 14. Recognition of Figures and Groups
"Welbent" Series, Stage 1, Step 1.

No special grouping was used on these cards, for it was thought best to let the child see the numbers in different aspects. Therefore in many cases it will not be a matter of quick recognition, but of counting. When filling in the groups, however, the child should be taught to use some definite grouping.

(*c*) The sorting occupations described in Chapter II, Section II, (2), are useful here if number-cards or digit-tablets are supplied to be placed beside each group.

(3) **Putting Number-cards or Digit-tablets on the Number-rods and Short Bead-bars.** The child will probably put the cards on the rods taken in any order. It should be suggested to him then to arrange them in ascending or descending order.

(4) **Exercises with the Wall-key and Test-board.** This board (see Fig. 15) is made of wood, and the figures, groups, words,

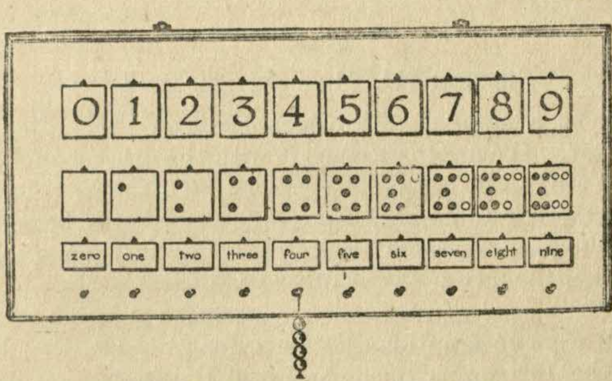


Fig. 15. Wall-key and Test-board

and bead-bars are detachable, so that the children can take them off and put them on in any order.

It is important for the teacher to see that they are never left up on the wrong pegs. They need not, of course, be in ascending order as in the diagram, but groups and word-boards should be placed under the right figures. The children should be taught to refer to the board when in doubt about the meaning of a symbol.

Opinions differ as to which grouping should be used. See Section I, (2), above.

The words may be removed when the board is being used by small children. It is, however, an excellent plan to let the words hang under the groups at other times for the children to get to know them as look-and-say words. If this were done more frequently in schools few children would have difficulty in spelling " two," for the word would be connected in their minds with the group and figure.

This test-board is very useful for simple group-games. All the figures and groups may be taken off and dealt out to the

children. Each child in turn replaces his tablet. The lowest line of hooks may be used for hanging up strings of beads or natural objects.

Sorting cards with pictures of natural objects with number of parts corresponding to the figures above (for instance, a clover leaf for three) provides an engrossing occupation for a little child.

Section III. Recognition of Zero

Zero is a symbol with which children very often have difficulty. We need not teach it with the other symbols, but might wait until the child needs it. On the key and test-board there is room only for ten figures—both the zero and the ten should be provided, for many teachers prefer to introduce the zero with the other symbols.

(1) This need of zero will arise when the child comes to formal work in notation and finds, for example, that he has 2 in the 100-compartment and 5 in the units, and nothing in the tens. Our word 'cipher' or 'zero' comes from the Arabic word meaning 'empty.' The empty space used to be represented by a dot—4.5. A circle was later put round the dot to make it more conspicuous; thus we got 4⊙5, which still later developed into our 405.

(2) The need may be aroused earlier, as Dr Montessori suggests, by putting the symbol 0 in one of the compartments of the trays described above. Her method of teaching it is best given in her own words—it is quite excellent.[1]

> We wait until the child, pointing to the compartment containing the zero card, asks "And what must I put in here?" We then reply "Nothing: zero is nothing." But often this is not enough. It is necessary to make the child *feel* what we mean by nothing. To this end we make use of little games which vastly entertain the children. I stand among them, and turning to one of them who has already used this material I say, "Come to me zero times." The child almost always comes to me, and then runs back to his place. "But, my boy, you came one time, and I told you to come zero times." Then he begins to wonder. "But what must I do, then?" "Nothing: zero is nothing." "But how shall I do nothing?" "Don't do anything. You must sit still. You must not come at all. Not any times. Zero times. No times at all."

I repeat these exercises until the children understand, and they are then immensely amused at remaining quiet when I call to them to come to me zero times, or to throw me zero kisses.[1]

(3) Dr Montessori suggests another game which tests the children's knowledge of figures and numbers up to ten. Each child draws a card with a number on it, puts it face downward on his desk, and then fetches the corresponding number of objects from the teacher's table. The child who has zero will have to remain at his place. The teacher then goes round to check the result.

II. PLACE-VALUE

A knowledge of place-value is needed to work almost every sum, and hence its importance cannot be over-estimated. It is not possible to deal with it in one chapter, for it has to be taught again and again in relation to the different arithmetical processes. In this chapter it will be possible only to discuss apparatus and to give a few suggestions for exercises in place-value as such. Many other exercises will be found in the chapters dealing with addition, subtraction, multiplication, and division.

In all the earlier written work the children should set down their sums in columns headed "H," "T," "U." It does not seem desirable, however, to make two stages of this—for instance, when using bundles of 10-sticks and loose sticks to teach the child first to put "b" for bundle and "s" for loose sticks, and then later to substitute "T" and "U" for "b" and "s." The important fact is not that the sticks are in bundles, but that they are in bundles *of ten*.

SECTION I. APPARATUS FOR TEACHING PLACE-VALUE

The following apparatus is recommended:

(1) Montessori number-cards (see Fig. 18), used with short bead-bars.

(2) Notation-trays with 100-boxes and short bead-bars (see Fig. 19).

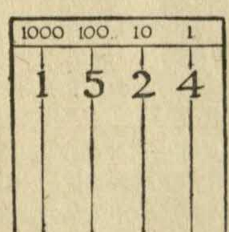
[1] *The Montessori Method* (Heinemann), p. 329.

(3) Long bead-bars—with tags (see Fig. 34).
(4) Coloured counters (see Fig. 46).
(5) Cards ruled in columns and headed " U," " T," " H," " Th," etc. (see Fig. 21).

This apparatus will be described in the sections dealing with the exercises for which each is used.

The Abacus. In the formal abacus the units, tens, hundreds, etc., are distinguished from one another by the beads on each wire being of a different colour. Hence it is not only *position* that indicates value, but another arbitrary quality—colour.

Montessori Abacus.[1] There are two of these so-called number- or counting-frames. One has four and the other



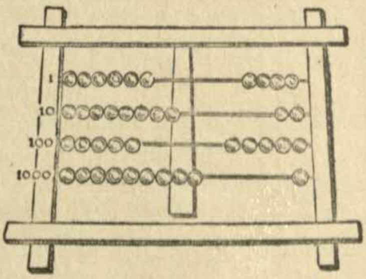


Fig. 16. Montessori Abacus

seven wires. The units, tens, hundreds, and thousands are represented by green, blue, red, and green beads respectively. The abacus stands up vertically ; hence the wires have to be horizontal. Thus we get the unit line at the top, with the 10-wire immediately below it (see Fig. 16). In writing down the figures the child has to put on the left the figure representing the number of beads on the fourth or 1000-line, and the number for the third or 100-line on the right ; the tens on the right of that, and the units on the extreme right.

To facilitate this process Dr Montessori has devised what seems an unnecessarily complicated arrangement. She has paper specially ruled with four vertical lines, one each of green, blue, red, and green, corresponding to the green, blue, red, and green beads on the units, tens, hundreds, and

[1] For details see *The Advanced Montessori Method*, vol. ii, p. 204. For the history of this abacus see Benchara Branford's *A Study of Mathematical Education*.

thousands wires. The child has to write on the green line on the left the number of beads on the 1000-wire.

The Horizontal Abacus (Fig. 17).[1] The advantage of this abacus has been realized by Dr Jessie White. By having the frame horizontal, and the child's paper the same width, the units column will come under the units wire.

An abacus is undoubtedly useful in *explaining* place-value, but it has grave disadvantages when used for working sums.

The chief of these is that sums of the type 28 + 37 cannot

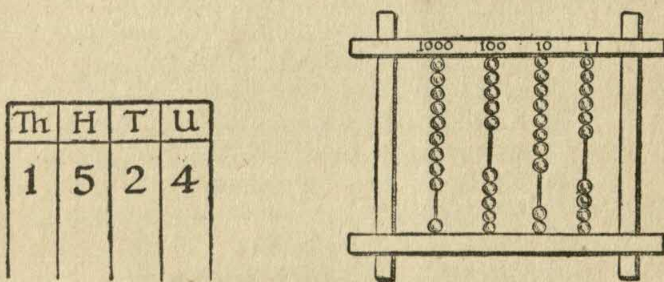


Fig. 17. HORIZONTAL ABACUS
To lie flat on the table.

be set out *complete* and *worked straight through* as when trays and beads or counters are used.

In working the above sum, the child

(1) Slides down eight beads on the units wire, then two more, *remembering* that five of the seven still remain to be added—thus the numbers are, as it were, *split up*.
(2) Slides back the ten units, and moves down one bead on the 10-wire instead.
(3) Slides down five units to complete the seven.
(4) Slides down five beads on the 10-wire and gets the total of 65.

If a child is advanced enough to say 8 + 7 = 15, then the use of the abacus seems quite unnecessary. Its chief value is undoubtedly for such exercises as bring out the *principle of notation applied to high numbers*. Even here children have been observed to work the sum in their heads and then

[1] See footnote on p. 60.

represent the answer on the abacus, showing clearly that when once the principle is understood the abacus has little value. Indeed, its prolonged use tends to make children count in ones and work 'through the ten.' Counters in seven or eight different colours would serve the same purpose, and show results just as clearly. (See Fig. 46.)

Section II. Exercises introducing Place-value

(1) **Counting.** Place-value is perhaps best introduced by exercises in counting. In the chapters on the four rules a series of such exercises will be given, and, as has been said, it is by watching a child perform these exercises that a teacher will be able to detect the best moment for giving a first explanation of place-value. When this moment arrives the following suggestions may be found useful.

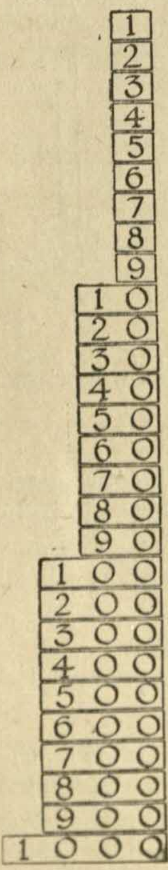


Fig. 18
Montessori Number-cards

Special Counting Exercises. (*a*) The child counts up the long bead-bar, saying:

"10 is 1 ten and no more."
"11 is 1 ten and 1 more."
"12 is 1 ten and 2 more."

"20 is 2 tens and no more."
"21 is 2 tens and 1 more."

(*b*) The teacher marks off a number of beads—for instance, 32—and the child says: "Three tens and 2 more make 32."

Further counting exercises then follow, in which tags and numbers are fixed on to the bar. (See Fig. 34.)

(2) **Series of Numbers. Exercises with Montessori Number-cards** (see Fig. 18). (*a*) The child lays the cards out as in Fig. 18, placing the corresponding bead-bar beside each card. The sequence of the numbers is thus clearly shown. The cards may also be laid out in three columns—1–9, 10–90, 100–900.

(*b*) By placing one card upon another the child can represent any number asked for up to 1000 ; *e.g.*, by placing the 3-card on the 900-card he makes 903.

(*c*) Similar number-cards made on a smaller scale might be used with the notation-tray instead of digit-tablets, but it must be remembered that with them the difficulty with regard to the zero is anticipated.

(*d*) It is a good plan to print the name of each number on the reverse side of the cards. After a child has built up a number he can turn the cards and see the building-up of the words.

(3) **Exercises which should be worked with Notation-trays, Number-cards, and Small Bead-bars.** The child is required :

(*a*) On *hearing* the name of a number to put the corresponding number of bead-bars into the tray.

(*b*) On *being given* a number of bead-bars to *place* them in the tray and to *say* the name of the number.

(*c*) On *seeing* the written symbol of a number—*e.g.*, 32 —to *find* and to *place* the corresponding number of bead-bars in the tray and to *say* the name of the number.

(For these exercises cards with the *whole number* written on should be used. The child then, as it were, analyses the number into tens and units.)

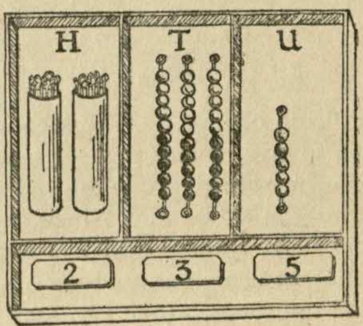


Fig. 19

(*d*) On *being given* a number of bead-bars to *place* them in the tray and to find the digit-tablets which, when put together, will form the number. In this case the child builds up the symbol from the number of beads. (See Fig. 19.)

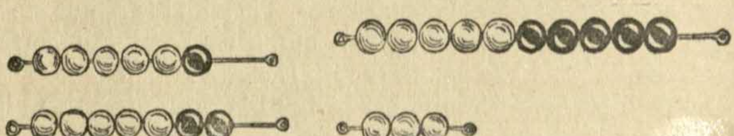
(*e*) On *being given* bead-bars to place them in the tray and to write figures or place digit-tablets on cards marked out in columns headed " Th," " H," " T," " U," to represent the number. (See Fig. 21.)

(*f*) To write any number, or place digit-tablets on paper marked out in columns, and then to represent that number in bead-bars in the tray.

The above exercises might also be worked with coloured counters.

The notation-trays can be made from boxes. For tens and units only cigar-boxes make excellent trays. The 100-boxes are easily made by covering cylindrical mantle-boxes, which hold exactly ten 10-bars. All the 100-boxes need not be open.

(4) **Game of "Please Change."** When first introducing children to place-value it is very helpful to teach them to play "Please Change" with the small bead-bars. The first child takes any two bead-bars under ten—for instance, a 7- and a 5-bar—and puts them in front of a second child, saying



First child: "Please change" (*putting down the 7- and 6-bars*). *Second child:* "1 ten and 3—13" (*putting down a 10- and a 3-bar*).

Fig. 20. Game of "Please Change"

"Please change." The second child counts the beads, and has to give in exchange as many 10-bars as possible. In this case he will give one 10- and one 2-bar. If the first child gives the 8-, 6-, and 7-bars, the second child would have to give two 10-bars and a 1-bar Children get astonishingly quick at changing, and very soon cease counting and simply add in groups. (See Fig. 20.)

Section III. Multiplying and dividing by Ten, Number-dictation, and Chart-making

(1) **Exercises with Coloured Counters.** Counters of four colours are provided—*e.g.*, yellow for units, red for tens, green for hundreds, orange for thousands. It is explained to the child that ten yellow are equivalent to one red, and ten red to one green, and so on.

(*a*) Children might then

(i) Place counters opposite groups of bead-bars.

(ii) Build up numbers with Montessori number-cards, and then interpret them in counters.

(*b*) The teacher might dictate a number, and the child put out counters on cards ruled in columns headed "Th," "H," "T," "U."

These exercises should not be multiplied, for once the child understands the principle underlining notation he should work without apparatus.

It is, of course, possible to extend these exercises to millions —by having more counters of different colours. Two or three examples involving millions might be worked, but to do more is likely to lead to waste of time. See notes on the abacus on pp. 60–62.

(2) **Multiplying and Dividing by Ten.** As a preliminary exercise the child might work with the notation-tray.

Starting with one 1-bar in the units place, the child should multiply it by 10, and then by 10 again—changing it for a 10-bar and then for a 100-box, and putting the corresponding number-cards below.

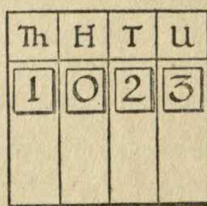


Fig. 21

Then the same process should be repeated with the 2-, 3-, and 4-bars.

This should be followed by exercises in the reverse process—viz., dividing 100, 200, 300, and so on by 10.

Counters might then be used to interpret his work with the bead-bars, and finally only digit-tablets and notation-cards, as shown in Fig. 21.

When numbers up to 9000 have been practised a card with columns to one million might be used, and the child might start with numbers such as 104.

The chief points to insist on are that the figures must move one place to the left when multiplied by ten, and one place to the right when divided by ten. At first little children should not be allowed simply to add a nought when multiplying by ten, but should rewrite the whole number, moving the figures to the left.

(3) **Number-dictation.** Words and symbols should be connected by the teacher dictating a number in words, while the children represent it on their notation-cards (Fig. 21) with digit-tablets. The child should be allowed to fill in first the

figures named—*e.g.*, for 202 he would put a 2 in the hundreds place and another 2 in the units place. Then he would realize the necessity of adding a zero in the tens place. At

1	10	100	1000	
2	20	200	2000	
3	30	300	3000	
4	40	400	4000	
5	50	500	5000	
6	60	600	6000	
7	70	700	7000	
8	80	800	8000	
9	90	900	9000	

Fig. 22

first the numbers dictated should have some connexion with one another, for when the digits are the same the value of place or position is accentuated; *e.g.*,

2	20	220	202
33	30	303	300

Any mistakes should be rectified by using bead-bars and notation-trays.

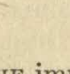
Children usually find most difficulty in writing and reading numbers with tens or hundreds of thousands or millions. A paper ruled in nine columns—the three on the left being headed "Millions," the central three "Thousands," and the right-hand three "H," "T," "U"—helps them very much. It also makes clear where they should put commas, and explains their importance and significance.

(4) **Making Charts.** A very useful exercise for children *who can write with ease* is to let them fill in a chart ruled in columns as in Fig. 22.

CHAPTER IV

ADDITION

THE importance of the child's acquiring from the very beginning good habits in adding cannot be over-emphasized.

Many people suffer throughout their lives through using bad methods of addition. Nearly all these methods are based upon the lengthy and laboured counting of units. Many of them are relics from previous generations of teachers who laboured under the vicious 'payment by results' system, and consequently were obliged to resort to mechanical devices to produce mechanical accuracy. And, unfortunately, these methods survive in some schools, and the poor scholar finds himself encumbered with clumsy props and methods which he seldom manages to discard. Such are the methods of 'tapping' or 'dotting' each figure and counting dots or taps; of ornamenting each figure with tails thus  and counting the tails singly; of putting down strokes at the side and counting and crossing off each stroke; of performing a tattoo with right or left hand or both, on table, knee, forehead or nose, gravely counting units as before. All these clumsy devices, seen all too frequently even among adults, are habits which *need not have been acquired*. . . . The formation of many of these bad habits is undoubtedly due to the attempt to do formal *written* sums in addition too early in the child's life before the necessary mental facility has been acquired.[1]

SECTION I. APPARATUS

The following are recommended for teaching addition:

(1) Number-rods.
(2) Reversible counters.
(3) Long and short bead-bars.

(1) **Number-rods.** The Montessori numerical rods and the 1-inch number-rods are described in Chapter III, Section I, (4). If neither of these can be provided strips of strong cardboard

[1] F. F. Potter, *The Teaching of Arithmetic* (Pitman).

divided into 1-inch sections should be used. Rods or cards divided into centimetres are too small. Strips of cardboard with *each section* numbered are also to be avoided, as the rows of numbers confuse the child, and lead him to think, for instance, that the second section is two, and the third section three. It is useful, however, to have the number of each strip marked on it on one side only and in the last section. For a long time the child will 'count through' every equation; this is not only helpful to his development, but is inevitable, as it is impossible for him to recognize at a glance a bar with more than three sections. This is because the sections are in a straight line and form no pattern. Even grown-ups will find it impossible to distinguish the 7- or 8-bars at a glance. The child should not be hurried over this counting stage, but by degrees he will use the figures more and cease counting the sections.

(2) **Reversible Counters.** Red and white, or red and black, reversible counters are recommended as being less distracting and forming clearer groups than the many coloured counters so often supplied.

(3) **Long and Short Bead-bars.** The various kinds of short bead-bars are discussed in Chapter III. The horizontal bead-bar with thirty large beads shown in Fig. 34 and smaller ones for each child are also very useful.

SECTION II. BEGINNINGS IN ADDITION

All teachers now agree that addition should be introduced in equation form. This has the following advantages:

(1) An equation can be read off as a sentence, or be set down as it is said; *e.g.*, $2 + 4 = 6$. "Two and four equal six."

(2) The rods, bead-bars, or other apparatus can be arranged so as to correspond with the order of the numbers in the equation.

(3) It avoids suggesting difficulties with regard to place-value for which the child is not yet ready.

Teachers, however, will not agree about the grading of the

work. Some will prefer to teach addition with numbers under 6 exhaustively first, and then to proceed to 9, 10, or 12.

In the "Welbent" Series, Stage 1 deals only with numbers up to 6, and Stage 2 with numbers 1 to 10. This course was decided upon because it is easy for teachers who prefer working straight from 1 to 10 to arrange the two stages as one longer stage. There is something to be said for both methods. If children are ripe for systematic work—that is, if they have had many and clear number-experiences in the nursery—at five and a half or six years of age they will readily deal with numbers 1 to 10 in one stage. On the other hand, it must be remembered that to read, express in concrete form, and work out an equation which is presented in writing is more difficult for a child than to 'make up a sum' with concrete material and then to express it as an equation. Hence it may be wise to allow children to build up equations involving numbers up to 10, but to give only the lower numbers in the first written equations presented to them for solution. All through the course with little children the practical work which gives experience and familiarity should be far in advance of the more formal work.

Nothing can be said in favour of stopping short at 9. Teachers do this to avoid the symbol for 10, which they think will puzzle the child. They want him to understand each step, and so they postpone dealing with the 10 until the child is ready to be introduced to place-value. This is a great mistake. The symbol for 10 should be presented without comment. Many children will have picked it up long before they have reached this stage. This is not slurring over a difficulty, but is a wise taking into account the mind of a child. He does not yet want to know why there should be two figures for the 10—he is not yet ready to understand it. Gradually the need will dawn upon him, and then the moment will come for a full explanation. We grown-ups are apt to over-analyse, and so to create difficulties for the child. Ten is too interesting and important a number to postpone, for has not the child ten toes and ten fingers?

Those teachers who advocate taking the child up to number 12 rather than only to 10 argue that there is a great advantage

in studying this number on account of its importance in our British weights and measures. However, the first thing to do is to get the child clear about decimal notation, and there seems a likelihood of his becoming confused if at this stage the same apparatus is used for decimal notation and for weights and measures. By adding number-rods of 11 and 12 inches or decimetres to the set of ten we may invite the stock difficulty of confusing 10 and 12 when 'carrying' in later sums. A child's first idea of 12 should be as 10 + 2. If 12 is to be studied it can be made by joining the 10-rod and the 2-rod, and this is the safer plan, for by it the importance of the 10-bar is emphasized.

Section III. Exercises in Addition

The following kinds of exercises should be made use of when teaching addition:

(1) **Building up Given Numbers with Apparatus, forming an Equation to correspond.** (*a*) *Working with Number-rods.* The child is provided with a set of number-rods and ten cards with figures 1 to 10 clearly printed on them. The card with figure 10 should be twice the size of the others. The child arranges the rods as shown in Fig. 9, and then he places the number-cards on their corresponding rods. He is then shown how, by placing the 1-rod beside the 9-rod, he can get a length equal to the 10-rod. He then says:

	"9 and 1 equal 10."
And similarly:	"8 and 2 equal 10."
And	"7 and 3 equal 10."
And	"6 and 4 equal 10."

By moving the 5-rod along, or by using a second 5-rod, he also finds that "5 and 5 equal 10." It is not advisable to introduce the multiplication sign here, for multiplication is best taught after the first steps in addition have been understood.

At first the work should be mainly oral. On no account should the child be required to write the whole equation. When some registration of results is thought advisable, result-

cards as shown in Figs. 23 and 24 should be used. The blanks should be filled in with digit-tablets. Later longitudinal strips of paper may be fastened on with elastic bands or paper-fasteners, so that the child has only to write the answers on the paper. If children are made to write too soon—that is, while writing itself is still a difficulty to them—*the arithmetic is sure to suffer.* They tire themselves in the effort to write well, and hence are less likely to grasp arithmetical connexions. Secondly, the writing makes so long an

Addition +	
9+	=10
8+	=10
7+	=10
6+	=10
5+	=10

Addition +	
1+	=10
2+	=10
3+	=10
4+	=10
5+	=10

Figs. 23, 24. Result-cards for use with Number-rods and Bead-bars

interruption that the sequence of the operations is broken, and they get their arithmetical connexions piecemeal instead of in sequence. Lastly, it is bad for the writing, which, for little children, should be an independent exercise, with good writing or figuring as its aim.

Special practice must next be given to

$$1 + 9 = 10$$
$$2 + 8 = 10$$

and so on. We cannot take it for granted that because a child knows that $8 + 2 = 10$ he also knows $2 + 8$. Moreover, it is more difficult to add a larger to a smaller number than *vice versa.*

The rods should be arranged again as in Fig. 9, and the 9-rod brought forward to the 1-rod, the 8-rod to the 2-rod, and so on. The result-card shown in Fig. 24 might be used.

Then by interchanging rods the child will be led to realize that $9 + 1 = 1 + 9 = 10$.

When the child has mastered the combinations of two numbers that make 10, the 10-bar may be removed, and similar exercises performed with the 9-bar as a check. Result-cards similar to those in Figs. 23 and 24 should be provided for all the numbers 4 to 10. As it does not really matter in which order the numbers are taken, one set of result-cards should do for a class of children.[1]

(*b*) *Working with Counters.* When counters are used it will be found best to begin by showing the child number-pictures. These he should already be familiar with from learning his figures (see Fig. 15). Starting, for instance, with four rever-

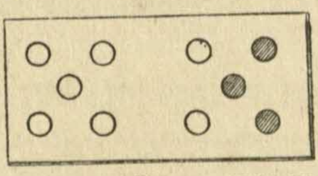


Fig. 25. WORKING THROUGH RESULT-CARDS IN FIGS. 23 AND 24 WITH REVERSIBLE COUNTERS—$7 + 3 = 10$, ETC.

sible counters arranged in a square, with red sides up, the child adds a black one, putting it in the centre, and so forms the equation $4 + 1 = 5$. When the combinations of 5 have been worked through, counters should be added one by one, so that each number is grouped as shown on the number-indicator (Fig. 15). The combination $7 + 3$ is shown in Fig. 25.

(*c*) *Working with Bead-bars.* The advantage of the "Wel-

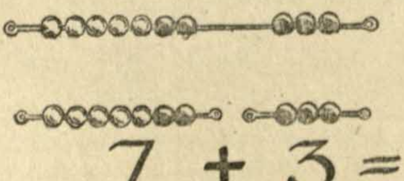


Fig. 26

bent" bead-bars, which are long enough to allow of the beads being separated, will be seen in Fig. 26, which shows

[1] Students are advised to read *The Psychology of Number*, by McLellan and Dewey, for notes on the Grube method. In this method lessons are given on number 1, then on 2, and so on up to 10.

how a single 10-bar may be used with or without two other shorter bars.

(2) **Given an Incomplete Equation, to read it and fill in the Missing Number.** (*a*) *Equations with Groups of Pips.* Fig. 27

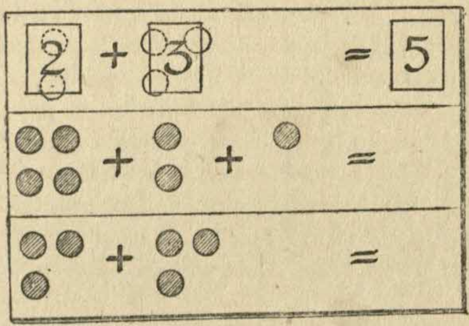


Fig. 27

Addition: Child covers each group with the corresponding digit-tablet, and then fills in the answer. "Welbent" Series, Stage 1, Step 5.

shows a simple form of card on which groups of pips have to be added together. It is important that a digit-tablet should be put on *each* group, so that when the equation is completed the child will be able to visualize the figures. Fig. 28 is a

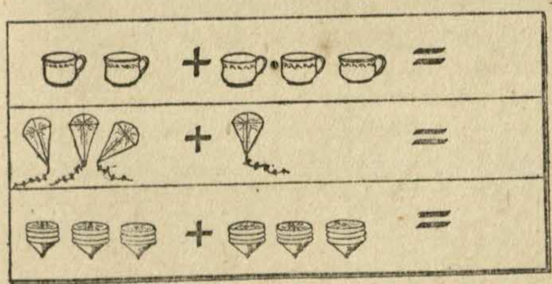


Fig. 28

A common form of card provided in schools, a drawback being that the variety of objects is apt to distract the child. The name of the objects should always be put after the answer; *e.g.*, "5 cups."

less desirable form of card, for there is so great a variety of objects that the child is apt to be distracted from the number-element involved.

(*b*) *Equations with the Left Side only given.* Such cards are shown in Figs. 29 and 30.

Many teachers will prefer the type with picture shown in Fig. 29 as being more concrete; it certainly is useful in helping the child to form problems.

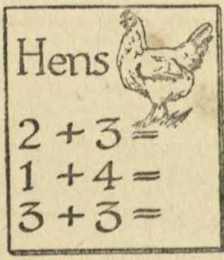


Fig. 29

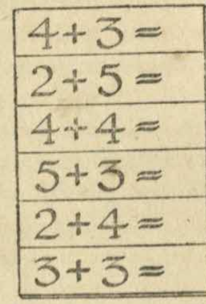


Fig. 30. Equation Card
"Welbent" Series, Stage 2, Step 8.

Both sets of cards may be worked through with rods, counters, or short bead-bars.

(c) *Equations with Blanks to be filled in on Right or Left Side.* Two forms of cards are shown in Figs. 31 and 32. The best way of working through the card shown in Fig. 32 for the

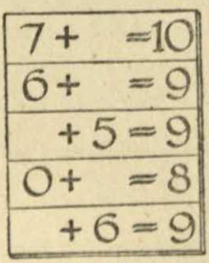


Fig. 31

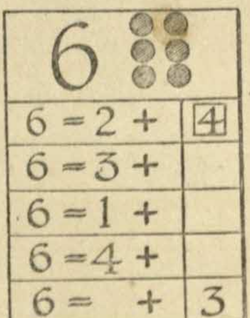


Fig. 32
"Welbent" Series, Stage 1, Step 7.

first time is by placing reversible counters, with red sides up, over the pips at the top of the card.

Then if the child reverses the number indicated by the figure on the right side of the equation the answer will be shown by the number of red counters remaining. The card might then be worked through again with short bead-bars.

The 6-bar should, of course, be taken for the card shown in Fig. 32.

(3) **Building up any Number with Concrete Material and then forming an Equation to correspond (working with Number-rods or Bead-bars).** In these exercises no set number is given for the right-hand side of the equation. There is, therefore, no check; hence the apparatus is not self-corrective.

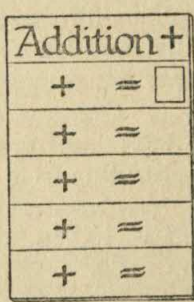


Fig. 33

Fig. 33 shows a form of result-card which is useful. Children will, of course, build up numbers higher than 10, and this they should be encouraged to do.

Section IV. Exercises with Numbers up to 30 or 50

When children have mastered the composition of all numbers up to 10, and have, as it were, 'made excursions' into higher numbers in their counting and building-up exercises, they have passed the first landmark.

The second landmark is reached when they have grasped place-value for tens and units, and have worked with numbers

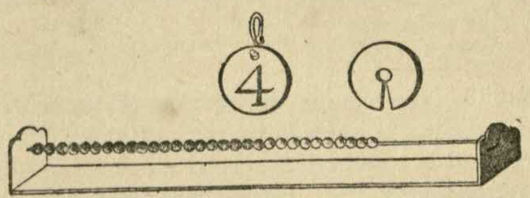


Fig. 34. The 30-bead Bar and Tag and Split Counter

up to 30 or 50. It is a pity to stop short at 20, for this gives the children no chance of dealing with any figure except a 1 in the tens place.

The following exercises are suggested.

(1) **Counting on the Long Bead-bar.** Fig. 34 shows the horizontal bead-bar with thirty beads. Dyed cotton-reels on an iron bar do equally well if beads are too expensive. Besides these big bars there should be smaller ones with thirty glass beads—ten of each of three colours.

Exercises might consist in :

(*a*) *Counting up and down the bar* for numbers above ten, saying, for example, " Eleven, 1 ten and 1 more," " Twelve, 1 ten and 2 more," and so on to " Twenty, 2 tens," " Twenty-one, 2 tens and 1 more."

(*b*) *Counting up and down in Twos, Threes, Fours, and Fives.* " 2, 4, 6, 8, 10, 12," and so on.

Starting from 3, " 3, 5, 7, 9, 11," and so on.

(*c*) *Fixing on Split Counters or Numbered Tags.* " Mark off 6, 12, 18, 24, 30." " Mark off 7, 10, 13," and so on up to 30.

(*d*) *Fixing on Split Counters or Numbered Tags and filling in Figures on Card with Two Columns headed " Tens " and*

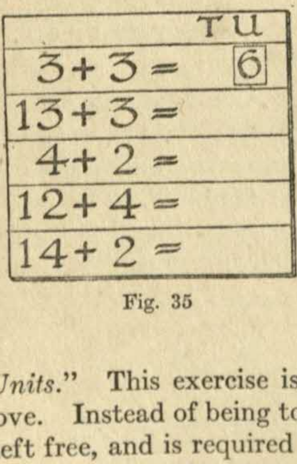


Fig. 35

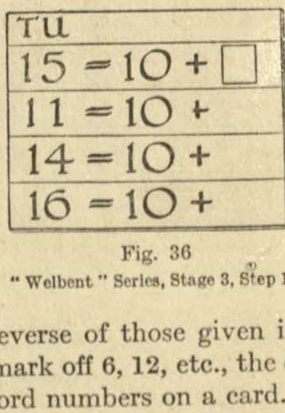


Fig. 36

" Welbent " Series, Stage 3, Step 1.

" *Units.*" This exercise is the reverse of those given in (*c*) above. Instead of being told to mark off 6, 12, etc., the child is left free, and is required to record numbers on a card.

(2) **Building up Numbers, using Short Bead-bars or Number-rods and forming Corresponding Equations.** For suggestion see Section III.

(3) **To complete Equations.** The work might be graded as suggested below.

(*a*) *Keeping ' within the Ten.'* Fig. 35 shows a card with equations involving addition. None of the equations involve ' crossing a border-line '—they require only addition of units. It is important, however, to emphasize the relations such as $3 + 3, 13 + 3, 23 + 3$. Fig. 36 is merely an ' interpretation ' card.

(*b*) *Adding in Tens.* With short bead-bars the child should build up series such as 10, 20, 30, 40, and so on; 3, 13, 23, . . . ; 7, 17, 27 . . .

(*c*) *Adding Numbers that complete Tens.* For instance, equations such as

$$8 + 2 =$$
$$18 + 2 =$$
$$28 + 2 =$$
$$16 + 4 =$$

And so on.

(*d*) *Adding Numbers involving 'crossing a Border-line.'* Fig. 37 shows a card in which the work is done by completing

	T	U
$7 + 5 = 10 + \boxed{2} =$	$\boxed{1}$	$\boxed{2}$
$9 + 3 = 10 + \boxed{2} =$		
$4 + 8 = 10 + \boxed{2} =$		
$8 + 3 = 10 + \boxed{1} =$		
$6 + 6 = 10 + \boxed{2} =$		

Fig. 37

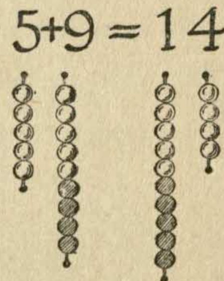


Fig. 38. Showing how more Difficult Combinations should be dealt with in Bead-bars

the ten. The intermediate step should be suppressed as soon as possible.

Fig. 38 shows an equation worked with bead-bars.

The game of "Please Change" should be introduced here. It is described in Chapter III.

Section V. Exercises on Numbers up to 100

The next great landmark is reached when the child can deal with numbers up to 100.

The first thing to do is to get the child to realize the magnitude of 100. For this no apparatus is better than short bead-bars, Dr Montessori's number-cards, and either her 100-bead chain (Fig. 39) or Dr Jessie White's 100-rod chain.

Fig. 39. A 100-CHAIN

Suggested Exercises. (1) *With Number-cards and Short Bead-bars.* (*a*) The number-cards up to 100 are put out as shown in Fig. 18. The corresponding number in bead-bars is then put next each card.

(*b*) Numbers such as 28, 38, 48, are built up by putting the card with unit 8 successively over the

Card 1	
90+	= 100
80+	= 100
70+	= 100
60+	= 100
50+	= 100
40+	= 100
30+	= 100
20+	= 100
10+	= 100

Fig. 40

Card 2
54 + 10 =
54 + 30 =
28 + 10 =
28 + 20 =
37 + 20 =
17 + 30 =
48 + 40 =
48 + 20 =
19 + 30 =
29 + 20 =

Fig. 41

Card 5	
95 +	= 100
85 +	= 100
75 +	= 100
65 +	= 100
55 +	= 100
45 +	= 100
35 +	= 100
25 +	= 100
15 +	= 100
5 +	= 100

Fig. 42

Card 6
96 + 4 =
86 + 4 =
76 + 14 =
66 + 4 =
66 + 14 =
86 + 14 =
56 + 14 =
24 + 6 =
34 + 16 =
54 + 16 =

Fig. 43

Card 10	
93 +	= 100
63 +	= 100
43 +	= 100
13 +	= 100
53 +	= 100
37 +	= 100
77 +	= 100
27 +	= 100
87 +	= 100
83 +	= 100

Fig. 44

This card gives practice on the combination 7 + 3.

zero on the 20-, 30-, 40-cards. The child may be left free to build up what numbers he likes.

(2) *With the* 100-*bead chain* (see Fig. 39[1]). By joining ten 10-bars together a chain of one hundred beads may be made. With separate 10-bars the child should work through a series of equation cards. As usual, before using these he should be

[1] A Montessori chain has beads all of one colour. As it is patented, chains with two colours cannot be bought at present.

left free to experiment and form his own equations. A few cards out of a graded series are shown in Figs. 40–44.

(3) *With the* 100-*rod chain.* This consists of ten rods each divided into 10 centimetres; alternate centimetres are coloured red. The rods may be folded so as to form a square.

When working Card 5 (Fig. 42) the advantage of bars showing fives will be realized. It is a simple matter to make a black mark at the centre of each rod.

The exercises will be the same as those with the 100-bead chain.

Section VI. Exercises with Numbers up to 500

The next landmark is reached when the child can add numbers up to a total of 500.

Except at first, to give him a realization of the magnitude

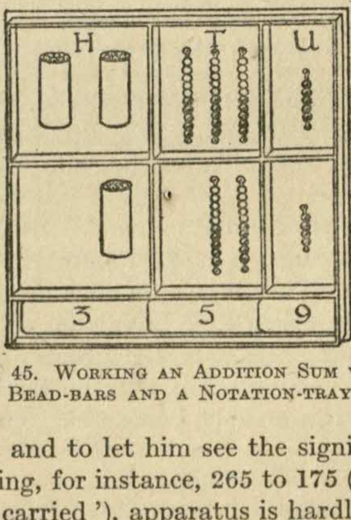


Fig. 45. Working an Addition Sum with Bead-bars and a Notation-tray

of the numbers, and to let him see the significance of place-value when adding, for instance, 265 to 175 (where a ten and a hundred are ' carried '), apparatus is hardly needed.

The child should begin to write his addition sums vertically, marking "H," "T," "U," at the top. Cards ruled in three columns in which digit-tablets may be placed are very useful.

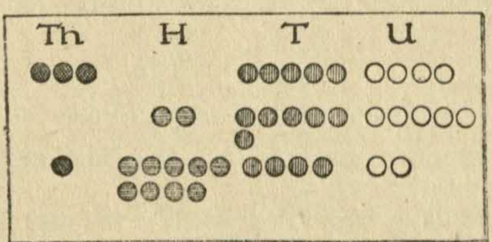
The notation-tray (Fig. 45), in which the hundreds are

formed by a chain, or by ten loose 10-bars being put into a mantle-box, is excellent for this stage.

The Montessori number-cards should also be used.

After this progress to numbers over 1000 should be rapid. An idea of 1000 may be given by putting out ten 100-chains, and allowing the child to handle them and make equations.

This is the stage at which an abacus might be introduced. See Chapter III, pp. 60–62.




```
ThHTU
 3054
  265
 1942
```

Fig. 46. Showing Arrangement of Counters for an Addition Sum
No special card is necessary, as counters may be arranged on the desk.

A *few* sums worked with coloured counters are very enlightening. And, as this method is almost indispensable for division, it is well to familiarize the child with it in sums on the other rules.

Fig. 46 shows a sum set out in bone counters of four colours —orange for thousands, green for hundreds, red for tens, and yellow for units. When a column adds up to more than 10, one counter of the colour next in value is substituted for the ten lower ones. In this way the process of ' carrying ' is made very clear, and there is not the bewildering moving backwards and forwards of beads that one finds in work with an abacus.

Section VII. Aids to Speed and Accuracy

(1) **Individual Work.** In the quotation given at the beginning of this chapter some pernicious habits were pointed out. By using suitable apparatus and by not expecting children to write too soon these may be avoided. However, there will

still remain a certain amount of drudgery to be gone through if any speed and accuracy are to be attained.

The habit—not only allowable, but positively helpful, in the first stages—of completing tens when adding must be dropped—or, rather, instantaneous recognition of the total of any two digits must take its place. For instance, on seeing

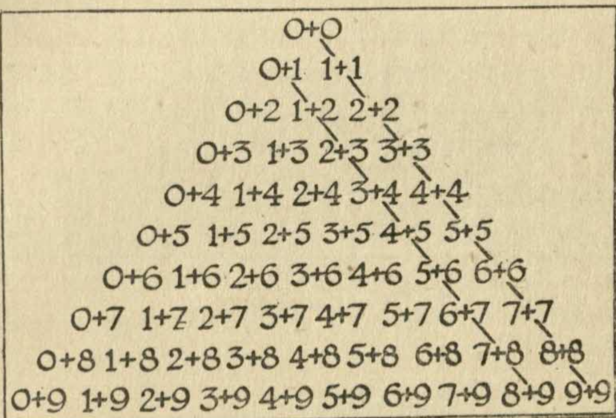


Fig. 47. CHART OF FIFTY-FIVE ADDITION COMBINATIONS TO BE LEARNT

The child should place tracing-paper over it, and write the answers over the combinations, working from 0 + 0 down to the right to 9 + 9, and then from 0 + 1 to 8 + 9, and so on.

7 and 5 the child should immediately say 12, and not have to work mentally thus : " 7 + 3 = 10 ; 10 + 2 = 12."

To achieve this much practice is required in the fifty-five combinations of digits.

Figs. 47 and 48 give the triangles of combinations in addition, and Fig. 49 shows how a small child might work through them with counters.

Fig. 50 shows an exercise in picking out and sorting the combinations according to their sum.

By putting transparent paper over the triangles the child might work through them, writing the answers on top of the figures and working to time.

The sums may be made more difficult by adding a 10 to each. The triangles would then include such combinations as 19 + 9, 18 + 8, 17 + 7, 16 + 6, 15 + 5, 14 + 4, and so on (see last line of Fig. 48).

9+0
9+1→8+0
9+2→8+1→7+0
9+3→8+2→7+1→6+0
9+4→8+3→7+2→6+1→5+0
9+5 8+4 7+3 6+2 5+1 4+0
9+6 8+5 7+4 6+3 5+2 4+1 3+0
9+7 8+6 7+5 6+4 5+3 4+2 3+1 2+0
9+8 8+7 7+6 6+5 5+4 4+3 3+2 2+1 1+0
9+9 8+8 7+7 6+6 5+5 4+4 3+3 2+2 1+1 0+0

Fig. 48 CHART OF FIFTY-FIVE ADDITION COMBINATIONS, BEST WORKED HORIZONTALLY

		0+0=0			
○	●	1+1=2		●	0+1=1
○○	●●	2+2=4	○	●●	1+2=3
○○○	●●●	3+3=6	○○	●●●	2+3=5
and so on to—		9+9=18	and so on to—		8+9=17

Fig. 49. SHOWING HOW A CHILD COULD WORK THROUGH THE CHART IN FIG. 47 WITH REVERSIBLE COUNTERS

			0+1	1	○
		1+1	0+2	2	●●
		1+2	0+3	3	○○○
	2+2	1+3	0+4	4	●●●●
	2+3	1+4	0+5	5	○○○○○
3+3	4+2	5+1	0+6	6	●●●●●●

Fig. 50. WORKING THROUGH THE CHART IN FIG. 47 WITH REVERSIBLE COUNTERS

Fig. 51 shows the two triangles combined to form a rectangle.

0+0	0+1	0+2	0+3	0+4	0+5	0+6	0+7	0+8	0+9
1+0	1+1	1+2	1+3	1+4	1+5	1+6	1+7	1+8	1+9
2+0	2+1	2+2	2+3	2+4	2+5	2+6	2+7	2+8	2+9
3+0	3+1	3+2	3+3	3+4	3+5	3+6	3+7	3+8	3+9
4+0	4+1	4+2	4+3	4+4	4+5	4+6	4+7	4+8	4+9
5+0	5+1	5+2	5+3	5+4	5+5	5+6	5+7	5+8	5+9
6+0	6+1	6+2	6+3	6+4	6+5	6+6	6+7	6+8	6+9
7+0	7+1	7+2	7+3	7+4	7+5	7+6	7+7	7+8	7+9
8+0	8+1	8+2	8+3	8+4	8+5	8+6	8+7	8+8	8+9
9+0	9+1	9+2	9+3	9+4	9+5	9+6	9+7	9+8	9+9

Fig. 51

Fig. 52 shows a book made by a child from strips of cardboard. Similar ones may be made in which the composition

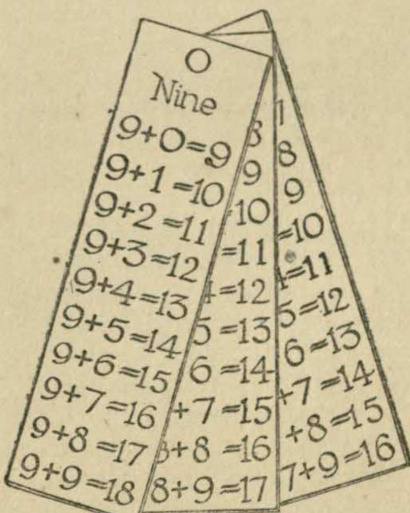


Fig. 52. Addition Book made by a Child

of each number is worked out; *e.g.*, $8 + 0 = 8$; $7 + 1 = 8$; $6 + 2 = 8$; and so on. In this case the strips will get

longer and longer as the numbers dealt with increase in magnitude.

Fig. 53 shows a specimen drill-card on the combination 7 + 5. Answers should be written on a strip of paper placed on the right of the card. Whenever a child is found weak in any combination he should work through a drill-card on that combination. It is essential that mistakes be traced to their origins, for very often inaccuracies are due to a weakness in some one combination.

7 +	5
17 +	5
17 +	15
37 +	5
57 +	5
87 +	5
97 +	5
7 +	15
7 +	25
7 +	55
17 +	25
17 +	15
27 +	15
37 +	25
5 +	17
25 +	17
15 +	17
35 +	7
45 +	17
65 +	7

Fig. 53. Drill-card in Adding

This should be worked to time. The child should be required to write the answer.

Small cards with the sum printed on one side and the answer on the other are very useful for children when memorizing. By going rapidly through a packet of such cards it is easy for a child to test himself and to note the combinations in which he is weak.

Some authorities maintain that the combinations are best mastered by practice in adding *three* digits—*e.g.*, 7 + 5 + 8. Numerous cards with such sums in vertical form should therefore be provided.

Magic Squares. A magic square is formed of numbers so placed that rows, columns, and diagonals have all the same total.

Very interesting exercises in addition and subtraction may be made by giving the child a chequered board and digit-tablets, and letting him (*a*) test given squares to see if they really are 'magic,' (*b*) complete magic squares in which some figures are fixed, (*c*) build up new magic squares from a given one. Figs. 54–61 show some magic squares. The figures indicate how new squares may be formed by adding, subtracting, or multiplying.

(2) **Oral Drill.** (*a*) In oral drill the habit of saying and thinking totals only should be cultivated; thus in the sum 3 + 4 + 7 + 8 the child should say only "3, 7, 14, 22."

Fig. 62 shows the time-honoured 'ring' of figures. By pointing to figures and signs in succession the teacher forms

	5	

Fig. 54
MAGIC SQUARE TO BE FILLED IN WITH DIGIT-TABLETS 1 TO 9, THE 5 BEING PLACED IN THE CENTRE

Rows, columns, and diagonals to add up to 15.

4	9	2
3	5	7
8	1	6

Fig. 55. SQUARE COMPLETED

Note that 15 equals 5×3—*i.e.*, 3 times the centre number. This holds for all squares of 3×3.

10	0	14
12	8	4
2	16	6

Fig. 56
SQUARE MADE WITH FIGURES 2, 4, 6, 8, 10, 12, 14, 16

14	4	18
16	12	8
6	20	10

Fig. 57. MADE FROM FIG. 56 BY ADDING 4 TO EACH NUMBER

11	5	8	10
2	16	13	3
14	4	1	15
7	9	12	6

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

Figs. 58, 59. FURTHER EXAMPLES OF MAGIC SQUARES

23	6	19	2	15
10	18	1	14	22
17	5	13	21	9
4	12	25	8	16
11	24	7	20	3

7	11	18	25	4
12	17	2	20	14
21	16	13	10	5
3	6	24	8	23
22	15	9	1	19

Figs. 60, 61. ARE THESE MAGIC SQUARES ?
If not, which figures are wrong ?

sums, which the children work mentally. It is sometimes useful to let a child work the sums aloud in order to secure correct habits.

(*b*) Counting in twos, threes, etc., beginning at 0 or at some higher number; *e.g.*, 0, 4, 8, 12, 16, 20, 24, counting in fours; 3, 7, 11, 15, 19, 23, 27, beginning at 3 and counting in fours.

(*c*) Counting by adding two numbers alternately—add 3, then 2; 0, 3, 5, 8, 10, 13, 15.

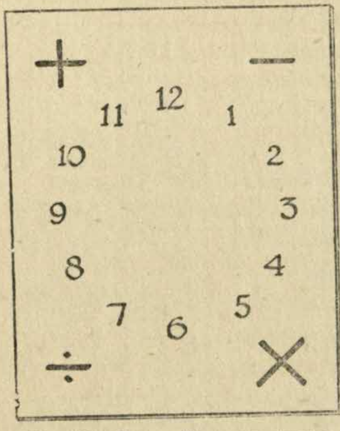


Fig. 62. Drill Number-circle

Section VIII. Formal Written Work

The following points are important:

(1) That the child acquire the habit of passing directly to the totals of two digits; *e.g.*, in the sum 15 + 6 + 4 + 16 the child should say mentally only 5, 11, 15, 21, for units column.

(2) That, if the figure to be 'carried' is put down at all (which is not necessary in short sums), it should be put at the top or bottom of the column to which it belongs.

(3) That all sums should be checked by adding from bottom upwards and a second time from top downwards.

(4) That it is a mistake to encourage a child to pick out addends that make ten. He will do this later of his own accord when the sum is short. In a long sum it leads to inaccuracies, as the child is apt to forget which addends he has used. It is best to train a child to add the figures as they stand, and to leave tricks and short cuts for later work.

(5) That horizontal as well as vertical adding should be practised.

(6) That in long sums it is important to train the child to take necessary rest at the end of a line or column. Some children are very prone to lapses in attention, and if these come in the middle of a line the answer to the sum will be incorrect. It is instructive to note whereabouts in a sum the inaccuracies occur. Frequently they will be found in the middle of the process.

The adding activity seems to give rise to nerve currents which have no habitual path of discharge, and so are dammed up, as it were, by the adding activity until the time comes when they are strong enough to throw the entire adding mechanism out of gear. They are then released and discharged along some nerve path in the body, causing involuntary movements, as sighs, frowns, etc., or altering the tension of certain muscles or the activity of certain internal organs, etc., etc. . . .

The degree of difficulty caused by such confusion currents varies enormously in different people. Some show no signs of it whatever. Some have an attention span of but three or four additions. There is evidence tending to show that the average span is about six or eight additions.

The difficulty is easily recognized by its symptoms. If the child hesitates at regular intervals in a column, if he goes over and over a given addition and is apparently unable to think at all, if he gives up in the middle of a column and begins again, the difficulty is almost sure to be one of this nature.

The remedy is obvious. Teach the child to recognize the difficulty when it occurs, to avert his attention momentarily by lifting his eyes and taking a deep breath, to keep his place in the column by pointing with his pencil, and to remember correctly the partial sum. Children are apt to get entangled in the situation, and to go over helplessly the same addition so long, that when the crisis occurs and the mind clears, the additions which had been made previously are totally forgotten.[1]

Section IX. Problems

A problem in arithmetic may require practical work such as drawing, measuring, and weighing. Such problems are discussed in Chapter IX. Very frequently, however, a problem is merely an application of number-facts to the events of everyday life, as, for instance, "Tom had 6 marbles. He lost 3. How many had he left?" Some authorities maintain that all the child's early number-work should be in the form of problems, or applications of number to everyday experiences. This is true for children under five and a half years of age. However, once formal number-work has been introduced it is a mistake for the teacher to present every example in problem form.

[1] *Courtis Standard Practice Tests, Teacher's Manual* (Harrap).

Children have a natural interest in numbers as numbers, and hence the approach to problem-work should be twofold.

(1) Through oral and written problems about familiar things presented to the child.

(2) Through the child weaving a problem round number-equations he has formed by using bead-bars and number-rods.

How to teach Problems. (1) Problems might arise out of games such as the train,[1] shop, and various floor games. In these no set problems are given, but the questions arise as the game proceeds. These games should lead naturally to written work—*e.g.*, when a shopkeeper makes out a bill for a customer.

(2) The problem may be stated orally and the child be provided with the actual objects mentioned, and may *act it.* "Tom had 5 marbles. He dropped 3."

(3) The teacher may draw on the blackboard, and let the child see the problem in picture-form—*e.g.*, "3 sparrows sat on a tree—4 more came."

(4) The child may be given beads and counters and be told to pretend that they are marbles, etc.

(5) By oral description the teacher may lead the child to realize the conditions of the problem, and to work it by means of small drawings, etc.

There is nothing to be said against these 'helps,' but it must be borne in mind that there is always a danger of their being over-done. For instance, it is a mistake for a teacher to think she must always weave a story round every series of sums, or that the children must be helped to visualize the objects in the problem by means of pictures, drawings, etc. A limited amount of this sort of work may be necessary, but it will become less and less so as more individual work with suitable apparatus is used.

The approach through the child's weaving problems around his equations is *extremely important.* Here the most difficult part of the work has been done, for the number-relations have been realized. Montessorians assert that a child who has

[1] For a detailed account of this game see the article on number in *Education by Life*, edited by Henrietta Brown Smith (published by Philip).

worked with the bead-bars, number-rods, and cards finds little difficulty in applying the knowledge to the solving of problems provided the conditions of the problem are within his ken.

Difficulties of Problems. If a child gets all his number-knowledge in the form of problems there is a danger of the words obscuring the number-relations. Also, it is more difficult to give him adequate practice in all number-combinations without wearisome repetition of sums on apples, marbles, nuts, etc., which are obviously made up for the sake of the numbers involved.

A written problem presented to the child involves not only reading, which is a difficulty in itself, but also his realizing the conditions of the problem. With regard to the reading, it would be far better for this type of problem-work to be postponed until the child can read the questions with ease. We should never present a child with more than one difficulty at a time. If the reading is a difficulty, then the conditions of the problem and the numbers must be easy. Yet it is for these last two that we set problems!

Early work in solving written problems may be simplified

(1) By using as few words as possible—*e.g.*, "1 yard costs 3*s*. 3 yards cost ?"
(2) By giving a simple picture of the object involved with the name printed beside it.
(3) By giving an easy introductory example when the numbers are high or where the problem involves two processes—*e.g.*, addition followed by subtraction.

CHAPTER V

SUBTRACTION

Section I. Beginnings in Subtraction

Subtraction should be begun as soon as a child has reached the first landmark in addition, for in a child's experience it certainly comes after addition. The two processes may be worked at together, the addition always being ahead of the subtraction, which is the more difficult process.

This difficulty is due to the fact that an example involving subtraction may be worded in a variety of ways, and that an equation is not so easily put into words. For instance, we have the following forms:

(1) "From 10 take 7," or "10 less 7." This is direct subtraction, and is stated as $10 - 7 = 3$.

(2) "What number must be added to 7 to make 10?" This is inverse addition, and may be stated as $7 + ? = 10$.

(3) "How much longer is the 10-bar than the 7-bar?" This implies comparison of the two bars, and may be worked as direct subtraction or inverse addition.

To become familiar with these forms a child needs a great deal of oral work. He must learn to adjust his apparatus to each form of question. For instance, working with the short bead-bars,

In (1) he takes the 10-bar and slides seven beads along, noting that three remain.

In (2) he takes the 7-bar and adds the 3.

In (3) he takes the 7- and 10-bars and compares them.

Some teachers advocate accentuating the idea of inverse addition. They allege that it is the more natural method, and easier for the child because so closely allied to addition.

Others say that the 'taking away' or direct subtraction is clearer for a child *just because* it is so different from addition. Children certainly find the 'taking away' method easy, and as the equation representing the process can be read straight through—10 − 7 = 3, "10 less 7 equals 3"—it is probably the best one to begin with. *Both* inverse addition and direct subtraction must be practised, no matter which formal method of subtraction (decomposition, equal additions, or complementary addition) is later adopted. Most adults use both processes mentally. When the two numbers are nearly equal in value they use inverse addition, but when one number is much larger than the other direct subtraction is found to be the quicker process. Thus in 106 − 87 inverse addition would probably be used, and in 106 − 17 direct subtraction. It is the *greatest mistake to start formal decomposition or equal additions before the child has grasped these mental activities.* For further discussion see Section V.

Section II. Exercises in Subtraction involving Numbers not Greater than 10

(1) Forming Smaller Numbers from a Larger One by 'taking away.' (*a*) *Working with Number-rods.* The child has put the rods together in his addition exercise so as to form a rectangle, the length of which is 10 inches. He has recorded the results as 9 + 1 = 10, 8 + 2 = 10, and so on.

He now proceeds to take one rod away from each combination. He may begin by removing the 1-rod, leaving the 9-rod, and record 10 − 1 = 9, or he may take the 5-rod away, leaving the second 5-rod, and record 10 − 5 = 5; 10 − 6 = 4; and so on. See Fig. 63 for a result-card which may be filled in from the top downwards or from the bottom upwards.

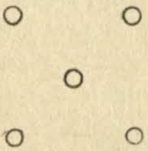
Take away —
10 − 5 =
10 − 4 =
10 − 3 =
10 − 2 =
10 − 1 =

Fig. 63

After this he could build up the tens again, and then take away the longer of each pair of rods, getting the following series of equations: 10 − 9 = 1, 10 − 8 = 2, and so on.

For further suggestions see corresponding paragraph on addition.

(*b*) *Working with Counters.* Here number-pictures should be formed. The child starts with, let us say, five counters arranged as here shown, all the red sides being up. He turns the central one to the black side, or he removes it, and then sees that he has four red counters left. He records 5 - 1 = 4.



(*c*) *Working with Bead-bars.* The child may work right through a set of bead-bars. Taking any one bar he likes—the order does not signify—he slides first one, then two, then three beads along (*i.e.*, he 'takes them away'), and records the results in equation form.

(2) **Given an Incomplete Equation, to read it and fill in the Missing Number.** (*a*) *Equations with Groups of Pips* (see Fig. 64). As both minuend and subtrahend are represented

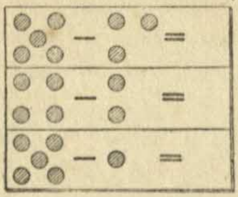


Fig. 64
"Welbent" Series, Stage 1, Step 6.

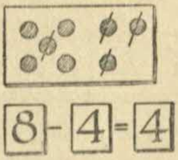


Fig. 65. Form of Card not recommended
Minuend printed in pips, subtrahend crossed out.

by pips the exercise involves comparison. Thus in the first example to make the 3 equal the 5 two pips would have to be added, and if the 3 of the subtrahend were placed on the 5 of the minuend there would be two uncovered pips. It is instructive to let children work these with counters. Where, as in the "Welbent" Series, Stages 1 and 2, a variety of grouping has been used, digit-tablets should be placed on each group.

The form of card shown in Fig. 65 is sometimes given to represent 'taking away.' The pips representing the subtrahend are already scratched out. The child has to form the equation. This type of card is not recommended.

(*b*) *Equations with the Left Side only Given.* Cards similar to those for addition (Figs. 29 and 30) should be used.

(c) *Equations with Blanks on Right or Left Side.* Figs. 31 and 32 show exercises involving subtraction by inverse addition. The children might weave problems round the equations on the card shown in Fig. 66.

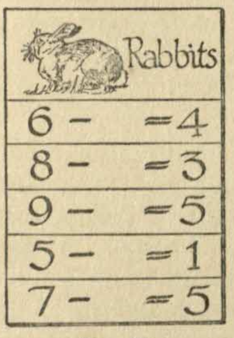


Fig. 66

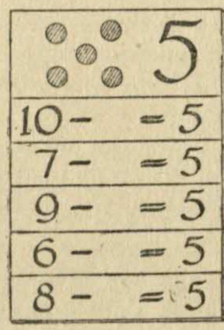


Fig. 67

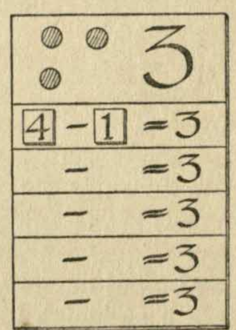


Fig. 68

The exercise in Fig. 67, if worked with ten counters arranged in two groups of 5 or with a 10-bead bar, is useful for emphasizing the relation of the lower numbers to ten.

If the child experiences any difficulty in using the card shown in Fig. 68 he should be invited to begin with four beads and work upwards—thus: $4 - 1 = 3$, $5 - 2 = 3$, and so on.

(3) **Breaking up any Number, using Concrete Material, and then forming Equations to correspond.** Fig. 69 shows a result-card which leaves the child free to form any equations he likes.

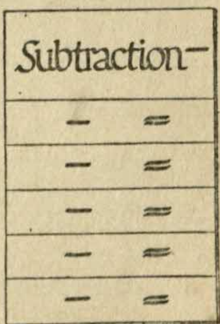


Fig. 69

Successive subtractions form a useful exercise; for instance, starting at 10 and subtracting 2 would give $10 - 2$, $8 - 2$, $6 - 2$, $4 - 2$, $2 - 2$.

Some children will spontaneously suggest filling in the corresponding addition-card (Fig. 33) at the same time. They will thus register two equations from one manipulation. On no account should a child be hurried into doing this, as it may lead to confusion unless he feels that he is ready for it.

Section III. Subtraction involving Knowledge of Place-value of Tens and Units, and of Numbers up to 30 or 50

When the child has mastered subtraction involving numbers under 10 he should be taught place-value for tens and units, and proceed with his *addition* until he has gained some familiarity with numbers up to 30. Then he is ready for more difficult subtraction.

In subtraction, so long as the equation form is used, as it should be at this stage, the child need not, and indeed *he should not*, be troubled by formal methods—*e.g.*, of decomposition, equal additions, or complementary addition. A discussion of the relative merits of each of these methods will be given at length in Section V. At the stage dealt with now, the child should do whichever manipulation corresponds to the question or exercise on which he is engaged. If only teachers would allow children to take their time and grasp thoroughly the ideas involved in subtraction, there would be little difficulty in teaching them equal additions or complementary addition later. The great thing is to get the child to express with suitable apparatus the process involved. By constant repetition the important combinations will become ripe for learning. A child who knows that $9 + 8 = 17$ will not necessarily know that $17 - 9 = 8$, but by suitable exercises he will come to realize the relationship, and then his progress will be rapid.

It shows bad preparation if, when a child is working a formal sum in subtraction, he finds himself held up at every step by not knowing at once, for instance, what $16 - 7$ equals, or, in other words, what he must add to 7 to make 16.

(1) **Counting on the Long Bead-bar.** (*a*) *Counting up and down the Bar in Twos, Threes, etc.* The exercises are on the same lines as those for addition (see Chapter IV, Section IV), but the emphasis is laid on counting backwards—that is, on successive subtractions.

(*b*) *Fixing on Split Counters or Numbered Tags.* "Mark off 30, 27, 24," and so on. "Mark off 22, then $22 - 4$," and so on, subtracting 4 each time. "Mark off 12, then $12 - 4$," and so on.

(*c*) *Fixing on Split Counters or Numbered Tags and forming Equations.* "Mark off 30, 28, 26, 24," and so on.

The child forms equations, as, for instance, "30 – 2 = 28," "30 – 4 = 26," and so on; *or* "30 – 2 = 28," "28 – 2 = 26,"

19 – 5 =
29 – 5 =
27 – 4 =
17 – 4 =
7 – 4 =

Fig. 70

"Welbent" Series, Stage 4, Step 6.

18 – = 10
26 – = 20
17 – = 10
27 – = 20
14 – = 10

Fig. 71

11 – 3 =
21 – 3 =
14 – 5 =
24 – 5 =
22 – 4 =
14 – 4 =

Fig. 72

26 – 23 =
18 – 15 =
23 – 21 =
23 – 11 =
14 – 12 =
24 – 12 =

Fig. 73

"Welbent" Series, Stage 4, Step 6.

and so on. He may mark "T," "U," over his figures or else use paper specially ruled.

(2) **To complete Equations involving Inverse Addition or Direct Subtraction.** The work might be graded as follows:

(*a*) Keeping 'within the ten' (Fig. 70).
(*b*) Subtracting so as to leave complete tens (Fig. 71).
(*c*) Subtracting in tens.
(*d*) Subtracting numbers less than 10, and involving the 'difficulty' or 'crossing a 10-line' (Fig. 72).
(*e*) Subtracting numbers greater than 10, without the 'difficulty' being involved (Fig. 73).
(*f*) Subtracting numbers greater than 10, with the 'difficulty' (Fig. 74).

By this time the child should be able to read an equation of the form 13 – 9 as direct subtraction or inverse addition ; hence it should not be necessary to multiply result-cards of the type 15 + ? = 20, shown in Fig. 75.

22-11=
28-18=
18- 9=
28-19=
25-17=

Fig. 74

Make 20
15+
12+
17+
13+
16+

Fig. 75

Fig. 76 shows a simple method of working an equation by inverse addition.

In Fig. 77, which shows the same equation worked by 'taking away,' it should be noticed that the 9 is not taken

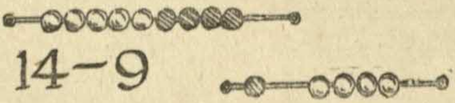


Fig. 76. BUILDING UP 9 INTO 14 BY ADDING 5

from the 10-bar only, but the 4-bar is combined with 5 from the 10-bar. In other words, direct subtraction has been used ; first 4 is taken away and then 5 more. The 10-bar is not broken into until there are no units left.

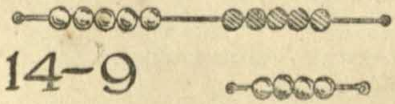


Fig. 77. 'TAKING AWAY' 9 FROM 14 AND LEAVING 5

As has been said in the last chapter, it is necessary to let a small child work 'through the ten' at first. In inverse addition the example 14 – 9 would be worked mentally thus : "9 + 1 = 10 ; 10 + 4 = 14 ; answer, 5."

In the 'taking away' method the reverse process would be the natural one—viz., "14 − 4 = 10; 10 − 5 = 5." Both these methods should be shortened by practice, so that the intermediate stage is left out. Unfortunately, a mixture of addition and subtraction is often taught; thus in the example 14 − 9 a child is taught to say "9 from 10 equals 1; 1 and 4 make 5"—*i.e.*, "10 − 9 = 1; 1 + 4 = 5."

A child accustomed to this usually finds it a difficult matter to learn his subtraction combinations, for he cannot shorten

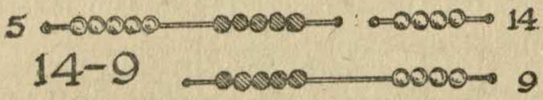


Fig. 78. Comparing 14 with 9

the process by simply leaving out a step. Another example may make this important point clearer. Suppose a child is asked to work mentally 45 − 8 and he does not know his combinations by heart. It is a most cumbersome way to proceed by saying "8 from 10 leaves 2; 2 and 5 make 7," and then adding the 7 to 30 to get 37. A much more straightforward method is to take 8 as 5 + 3, and pass through 40 to 37 direct. With practice this will soon be done in one step. The early teaching of formal decomposition, where the mixture of addition and subtraction is used, combined with the customary rigmarole of words, is quite fatal to rapid and clear work.

Fig. 78 shows this important point clearly.

Section IV. Subtraction of Numbers up to 100

The best pieces of apparatus for this are the 100-chain and short bead-bars.

Suggested Exercises. By laying single 10-bars along the 100-chain a series of exercises similar to those for addition may be worked.

The 100-chain may be folded so as to show, for example, 100 − 10 = 90. This is an advantage where bead-bars are not too plentiful.

The grading of the work might be as follows:

(1) Subtracting tens (*a*) from multiples of 10—*e.g.*, 90 – 10, 80 – 10; (*b*) from any number—*e.g.*, 95 – 10, 88 – 10.
(2) Subtracting fives from multiples of 5—*e.g.*, 100 – 5, 95 – 5.
(3) Subtracting without the 'difficulty' and with units only in the subtrahend—*e.g.*, 98 – 5; 97 – 4.
(4) Subtracting without the 'difficulty,' but with tens and units—*e.g.*, 98 – 17.
(5) Subtracting quantities which give zero in the units place—*e.g.*, 87 – 27.
(6) Subtracting any number from 100—*e.g.*, 100 – 63.
(7) Subtracting other numbers entailing the 'difficulty' —*e.g.*, 85 – 47.

(Figs. 40, 42, and 44 give cards with equations involving inverse addition.)

While the child is working all these exercises he should be encouraged to say as few words as possible. The example 85 – 47 worked with the 100-chain and short bead-bars will probably lead the child

(1) To take 40 from 85, and then 7 from 45, saying, "[85], 45, 38"; or
(2) To take 7 from 85, and then 40 from 78, saying, "[85], 78, 38"; or
(3) To build up 47 to 77, and then add 8, saying, "[47], 30, 38"; or
(4) To build up 47 to 55, and then add 30, saying, "[47], 8, 38."

Section V. Formal Methods

(1) **Discussion as to Relative Values of Decomposition, Equal Additions, and Complementary Addition.** By this time the child will have grasped the ideas underlying the processes of subtraction, and will also know most of his subtraction combinations. It is now necessary for him to learn to write his sums vertically, and to work rapidly by some approved method.

Equal additions or complementary addition as described below is recommended. Though such authorities as Miss Punnett and Miss Storr advocate decomposition, the weight of statistical evidence of the speed and accuracy of children in the upper school is in favour of either of the other methods.

Students are recommended to read *Mental Tests*, by Dr Ballard (pp. 168–174), also *Teacher's Book* to Part I of *Fundamental Arithmetic*.

In the following paragraphs it has been found necessary

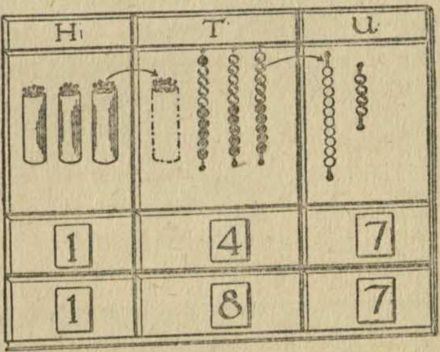


Fig. 79. Subtraction by Decomposition

Minuend in bead-bars, subtrahend in digit-tablets. (The ten 10-bars in the 100-box should be *emptied* into the tens place. This has not been done, in order to avoid complicating the diagram.)

for the sake of clearness to make the child say a rigmarole of sentences. As a matter of fact, a child should never say "7 from 4 I cannot," but should *see* the impossibility and act accordingly.

(2) **The Process of Decomposition** (see Fig. 79).

Example:

$$\begin{array}{r} 334 \\ -147 \\ \hline 187 \end{array}$$

The child may work in either of the ways indicated in (*a*) and (*b*) below, but ultimately the only words used should be at most "7, 14, 7; 5, 13, 8; 2, 3, 1."

(*a*) "7 from 4 not possible."

Take the 4 away. "Still three to subtract."

Take a 10-bar from the tens compartment. (These two steps may be used in reverse order.)

"3 from 10 leaves 7." Record 7.

"4 tens from 2 tens not possible."

Take 2 tens away. "Still 2 tens to subtract."

Take a 100-box from the hundreds compartment and *empty* it into the tens.

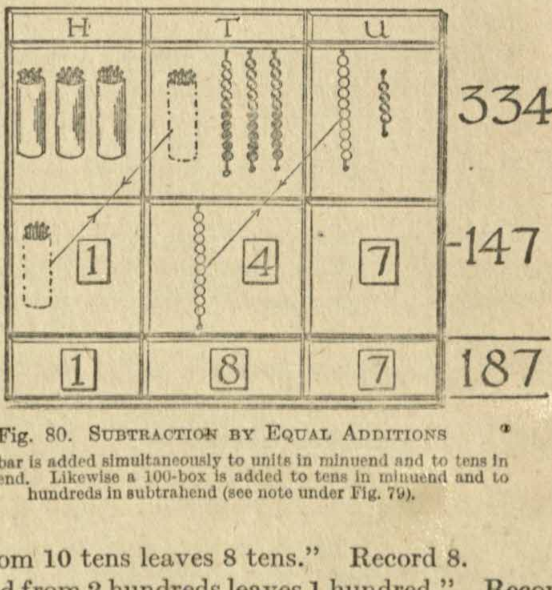


Fig. 80. SUBTRACTION BY EQUAL ADDITIONS

One 10-bar is added simultaneously to units in minuend and to tens in subtrahend. Likewise a 100-box is added to tens in minuend and to hundreds in subtrahend (see note under Fig. 79).

"2 tens from 10 tens leaves 8 tens." Record 8.

"1 hundred from 2 hundreds leaves 1 hundred." Record 1.

(*b*) "7 from 4 not possible."

Take 10-bar from tens compartment.

"7 from 14 leaves 7." Record 7.

"4 tens from 2 tens not possible."

Take 100-box and *empty* it into the tens compartment.

"4 tens from 12 tens leaves 8 tens." Record 8.

"1 hundred from 2 hundreds leaves 1 hundred." Record 1.

(3) **Equal Additions** (see Fig. 80). To introduce the idea of equal additions it is a good plan to get the children to work a few simple exercises to illustrate the fundamental law on

which this process is based—viz., " That if equals be added to unequals the difference will be unchanged." *E.g.*,

$$\begin{cases} 10 - 7 = 3. \\ (10 + 3) - (7 + 3) = 3. \end{cases}$$

$$\begin{cases} 9 - 7 = 2. \\ 19 - 17 = 2. \end{cases}$$

$$\begin{cases} 27 - 19 = 8. \\ 37 - 29 = 8. \end{cases}$$

Too much time should, however, not be spent over this —the children will understand the explanation better later on when they have used the process. Some children may grasp it at once; others will require the explanation to be renewed in a higher class.

In the example

$$\begin{array}{r} 334 \\ -\,147 \\ \hline 187 \end{array}$$

the process is as follows:

(*a*) " 7 from 4 not possible."

Add a 10 to the units in the minuend and to the tens in the subtrahend. " 7 from 14 leaves 7." Record 7.

" 5 from 3 not possible."

Add a 100-box to the tens in the minuend and to the hundreds in the subtrahend.

" 5 from 13 leaves 8." Record 8.

" 2 from 3 leaves 1." Record 1.

An intermediate method may be used—*e.g.*, " 7 from 4 not possible." Take the 4 away, 3 still remains to be subtracted from 10, which is decomposed to units.

(*b*) Here *mentally* the child works by complementary addition, but puts bead-bars out as in Fig. 80.

Add a 10-bar to the minuend units and a 10-bar to the subtrahend tens.

Then, " What must I add to 7 to make 14 ? " Record 7.

Add a 100-box to the tens in the minuend and to the hundreds in the subtrahend.

Then, " What must I add to 5 to make 13 ? " Record 8.
" What must I add to 2 to make 3 ? " Record 1.

(4) **Complementary Addition** (see Fig. 81). Here the subtrahend is built up into the minuend. (Counters might be used in the units instead of bead-bars if this is found easier.)

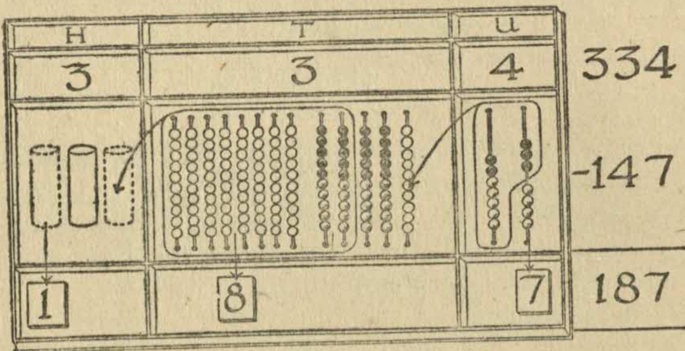


Fig. 81. SUBTRACTION BY COMPLEMENTARY ADDITION

Minuend in digit-tablets, subtrahend in bead-bars. The subtrahend is built up into the minuend.

The minuend is put out in digit-tablets, the subtrahend in bead-bars.

The example

$$\begin{array}{r} 334 \\ -147 \\ \hline 187 \end{array}$$

would be worked as follows :

(*a*) " What must I add to 7 to make 14 ? "

Add a 7-bar and record 7.

There are now 14 beads in the units compartment.

Change to a 10- and a 4-bar. Put the 10-bar into the tens compartment.

" What must I add to 5 tens to get 13 tens ? "

Add 8 10-bars and record 8.

There are now 13 10-bars in the tens compartment.

Change ten of them for one 100-box, and put it in the hundreds compartment.

" What must I add to 2 hundreds to make 3 hundreds ? "

Add one 100-box and record 1.

0-0
1-0 1-1
2-0 2-1 2-2
3-0 3-1 3-2 3-3
4-0 4-1 4-2 4-3 4-4
5-0 5-1 5-2 5-3 5-4 5-5
6-0 6-1 6-2 6-3 6-4 6-5 6-6
7-0 7-1 7-2 7-3 7-4 7-5 7-6 7-7
8-0 8-1 8-2 8-3 8-4 8-5 8-6 8-7 8-8
9-0 9-1 9-2 9-3 9-4 9-5 9-6 9-7 9-8 9-9
10-0 10-1 10-2 10-3 10-4 10-5 10-6 10-7 10-8 10-9 10-10

Fig. 82. SUBTRACTION CHART 1

18-9
17-9 17-8
16-9 16-8 16-7
15-9 15-8 15-7 15-6
14-9 14-8 14-7 14-6 14-5
13-9 13-8 13-7 13-6 13-5 13-4
12-9 12-8 12-7 12-6 12-5 12-4 12-3
11-9 11-8 11-7 11-6 11-5 11-4 11-3 11-2
10-9 10-8 10-7 10-6 10-5 10-4 10-3 10-2 10-1

Fig. 83. SUBTRACTION CHART 2

Drill-card on 8
10 − 2 =
12 − 8 =
15 − 8 =
20 − 8 =
23 − 8 =
25 − 8 =
30 − 8 =
36 − 8 =
32 − 8 =
44 − 8 =
41 − 8 =
47 − 8 =

Fig. 84

25-17	26-18	19-13	20-13	14-7	11-5
19-11	13-7	13-5	17-10	21-15	24-17
24-18	22-14	27-19	17-11	11-4	22-15
23-16	12-5	9-3	21-13	11-3	14-8
19-12	23-17	21-14	15-7	18-12	14-6
12-6	9-2	15-8	15-9	24-16	23-15

Fig. 85

Cover up all squares giving difference of 6 with green counters.
Cover up all squares giving difference of 7 with red counters.
Cover up all squares giving difference of 8 with blue counters.

Figs. 82 and 83 are intended to be used in the same way as the corresponding ones in Chapter IV on addition.

Fig. 84 shows one of a series of drill-cards that might be made to give practice in subtraction. The card shown deals with the subtraction of the number 8.

Fig. 85 shows a self-corrective type of drill-card. If the child has worked the equations correctly he will have a symmetrical pattern in coloured counters.

CHAPTER VI

MULTIPLICATION

SECTION I. MULTIPLICATION AS CONTINUED ADDITION

MULTIPLICATION should be taught, not as a new rule, but as a shortened form of addition. By means of examples such as $2 + 2 + 2 + 2 = 2 \times 4$, $3 + 3 + 3 = 3 \times 3$, the child will be led to see that multiplication is *continued addition of the same quantity*. The multiplication sign will then be accepted as a 'shorthand' form for expressing an addition sum. This conception of multiplication will, of course, have to be extended when multiplication of fractions is introduced.

Great care should be taken in the interpretation of the multiplication sign. 2×3 should be read as "Two multiplied by three," or "Two taken three times," or "Three times two."

Even though the *result* is the same if we read it as "Two times three" confusion results when concrete material is used. This question is dealt with again in Section III.

Opinions differ as to when a child should be introduced to multiplication. Dr Montessori teaches 5×2 with the number-rods, at the same time as addition combinations —*e.g.*, $7 + 3$. This seems to be a mistake. If multiplication be taken as continued addition it is wiser to postpone it until the child has had sufficient experience in adding to appreciate the shortened form of multiplication. When a child has had practice in such sums as $2 + 2 + 2 + 2 + 2 + 2$, $5 + 5 + 5$, $4 + 4 + 4 + 4$, he will better realize the advantage of expressing them as 2×6, 5×3, 4×4.

Moreover, when the multiplication sign is introduced before the child has mastered addition up to 12 there is not only little scope for practising a variety of examples, but also a danger of the plus and multiplication signs being confused.

On the whole it seems best to teach multiplication when the child has worked with numbers up to 20 in addition.

Section II. Beginnings in Multiplication

(1) The Meaning of the Multiplication Sign. (*a*) *Interpreting Addition as Multiplication.* Figs. 86 and 87 show exercise-cards in which simple examples in addition have to be expressed as multiplication and then the answer filled in. At

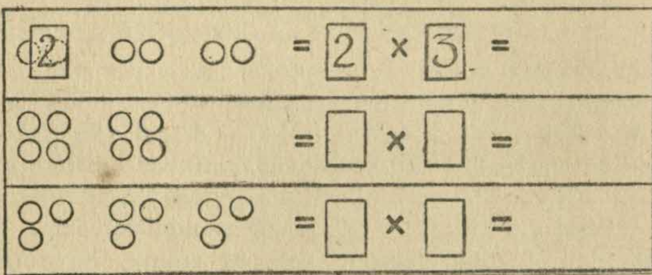


Fig. 86. Meaning of Multiplication

The child covers each group with a figure, and fills in the answer with digit-tablets—2 + 2 + 2, or 2 taken 3 times.
"Welbent" Series, Stage 3, Step 6.

first groups of pips are used, and in order to accentuate the repetition of the same quantity the child should put a digit-tablet on each group or write out the expression in figures with plus signs between—*e.g.*, $2 + 2 + 2 = 2 \times 3 = 6$.

2 + 2 + 2 + 2	=	2	×	4	= 8
2 + 2 + 2	=		×		=
3 + 3 + 3	=		×		=
4 + 4	=		×		=

Fig. 87. Multiplication

(*b*) *Interpreting Multiplication as Addition.* Here we start with the shortened form and expand it into the longer form. Fig. 88 shows a card which the child may fill in either with counters or with digit-tablets.

(c) *Exercises with the Addition and Multiplication-box.* The aim of these exercises is to give experience in using the multiplication sign and to ensure that the child realizes

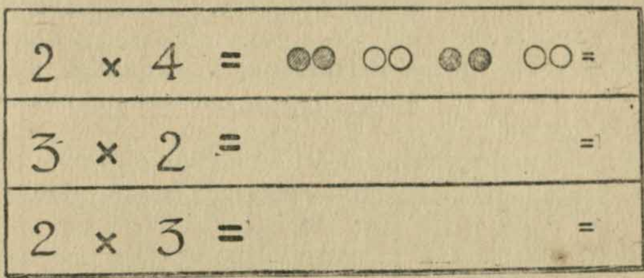


Fig. 88. Meaning of Multiplication
Child interprets 2 × 4 in counters of two colours.
"Welbent" Series, Stage 3, Step 7.

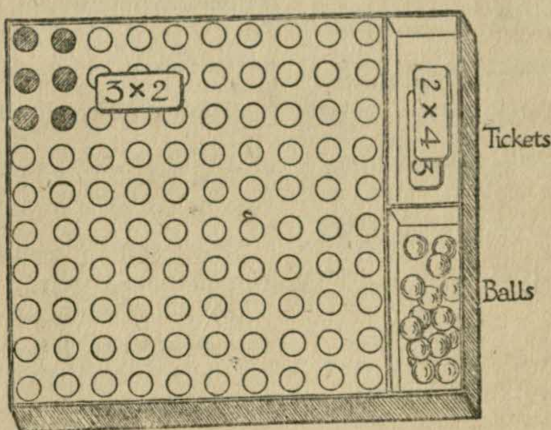


Fig. 89. Meaning of Multiplication
Perforated tray with coloured balls and tickets to interpret.
"Welbent" Series, Stage 3, Step 8.

that this sign cannot be used unless *equal quantities* are added.

Fig. 89 shows a box with a perforated tray on to which balls or beads are to be put to represent simple examples in addition and multiplication. In Step 8, Stage 3, of the "Welbent"

Series thirty sum-tickets are provided, arranged in three groups of graded difficulty. The child takes a ticket and represents the sum in balls, using a different colour for each number.

The reverse process is also necessary. The child or the teacher puts out a series of balls—*e.g.*, 2 green, 3 red, 4 white—and the child then writes the corresponding equation, $2 + 3 + 4 = 9$. Then 3 red, 3 blue, 3 white balls are put out, and the child writes 3×3.

This box is not recommended for building up tables. In this connexion see Section III, p. 112.

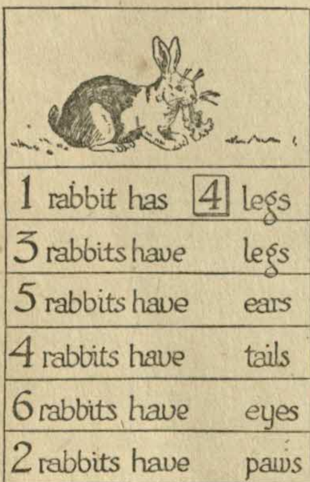


Fig. 90
Similar to "Welbent" Series, Stage 4, Step 8.

(A Halma, or other chequered board forms an admirable substitute if counters are used.)

(2) **Building up Numbers by Addition of Equal Groups.** (*a*) *With Long Bead-bar.* Examples such as the following might be given:

Mark off 3 beads 4 times.
" " 4 " 3 "
" " 2 " 6 "
" " 6 " 2 "

The result in each case is recorded immediately.

(*b*) *With Counters* (*or Multiplication-box*, Fig. 89). "Take 18 counters and group them in as many ways as you can. Write down results."

$$9 \times 2 = 18. \quad 6 \times 3 = 18. \quad 18 \times 1 = 18.$$
$$2 \times 9 = 18. \quad 3 \times 6 = 18. \quad 1 \times 18 = 18.$$

The last two equations are the most difficult for a child to grasp, and hence should come after the others have been mastered.

For further suggestions see Fig. 116.

(3) **Simple Problems.** Some teachers will prefer to postpone these until after the first tables have been built up. In any

case, the child should be encouraged to invent his own, and only when he has done this should he be introduced to cards of the type shown in Fig. 90.

It is usual in many schools to provide cards like that in Fig. 91. The danger here lies in the invitation to count in units. For some children such cards may be necessary in order to accentuate the repetition, but as a rule a picture of

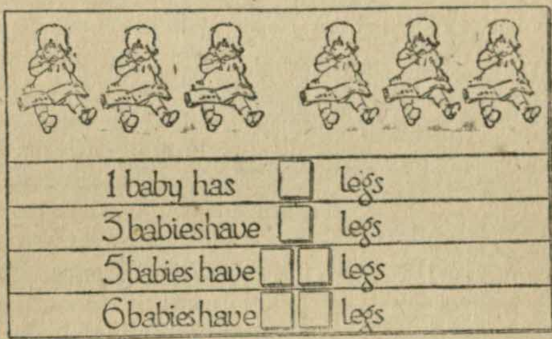


Fig. 91. MULTIPLICATION CARD

Allowable only for very small children, as it tends to make them count in ones.

one object only should be given, as this encourages the child to add in groups or to fill in the answer learnt from previous experience.

SECTION III. BUILDING UP TABLES

Opinions differ as to the best order in which to take the tables for the purpose of building them up. The following order is suggested: twos, threes, tens, fives, fours, sixes. In any case, they should be connected with one another as far as possible. For instance, the table of fours should be built up beside that of twos, and the threes and sixes should likewise be connected. (See Fig. 113.) At first only the form shown in the chart in Fig. 92—that is, the table of twos, threes, fours, etc.—should be used for building up, and it will be sufficient if the child builds as far as 2 × 6. Later the tables must be extended to twelve. The second form—that is, the two, three, and four times tables (Fig. 93)—might be

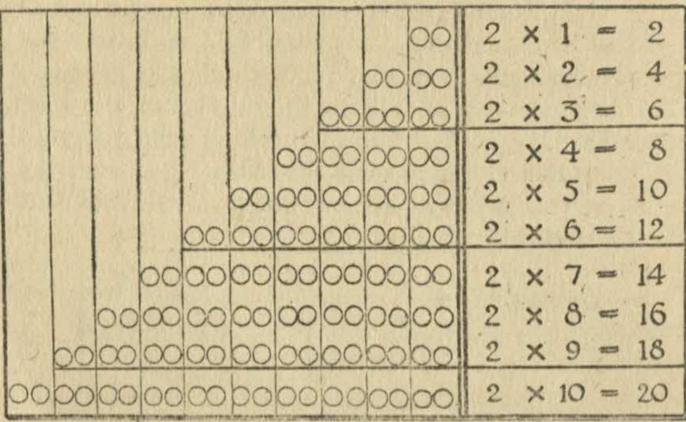


Fig. 92. MULTIPLICATION TABLES. TABLE OF TWOS CHART

This may be read as

2 taken once.
2 taken twice.
2 taken three times.
2 taken four times. Etc.

Or 2 multiplied by 1.
2 multiplied by 2.
2 multiplied by 3.
2 multiplied by 4. Etc.

Or One times 2.
Two times 2.
Three times 2.
Four times 2. Etc.

Or One 2 is 2.
Two 2's are 4.
Three 2's are 6.
Four 2's are 8. Etc.

Children find the first two the easiest. The last readings involve inverting —that is, beginning with the second figure—*e.g.*, 2 × 3, " *Three* times 2."

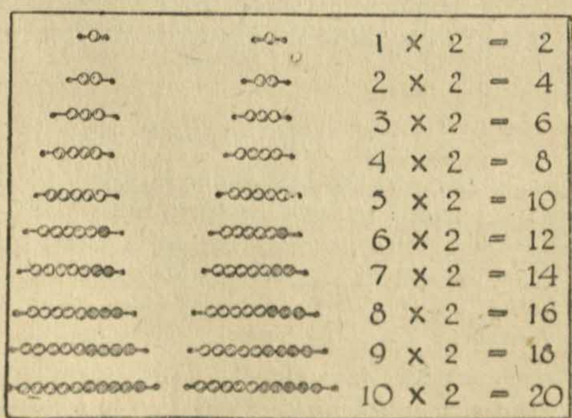


Fig. 93. MULTIPLICATION CHART OF TWO TIMES TABLE

This table may be read as

1 taken twice.
2 taken twice.
3 taken twice.
4 taken twice. Etc.

Or, inverting, 2 times 1.
2 times 2.
2 times 3.
2 times 4.

Or Twice 1 are 2.
Twice 2 are 4.
Twice 3 are 6.
Twice 4 are 8.

used later as an exercise. Fig. 94 shows a result-card for the table of threes, Fig. 95 one for the two times table ; a strip of pâper might be placed down the right side for the answers.

(1) **With the Long Bead-bar.** The table of threes, for instance, may be built up by fixing on tags and thus marking off the beads in threes. The results are then recorded on a card as in Fig. 94.

(2) **With the Short Bead-bars.** These are quite the best apparatus for building up tables, as they encourage the child to calculate in *groups.* Boxes for each table should be provided ; each box should contain twelve bars of the same number.

When a child is building up, let us say, the table of fours it is instructive to let him register his results in short bead-bars as well as on a result-card. For instance, 4×4 would be registered as one 10-bar and one 4-bar. This provides a revision in place-value and in the " Please Change " game.

$3 \times 1 = \boxed{3}$
$3 \times 2 =$
$3 \times 3 =$
$3 \times 4 =$
$3 \times 5 =$
$3 \times 6 =$
$3 \times 7 =$
$3 \times 8 =$
$3 \times 9 =$
$3 \times 10 =$

Fig. 94

Two times Table
$1 \times 2 = \boxed{}$
$2 \times 2 =$
$3 \times 2 =$
$4 \times 2 =$
$5 \times 2 =$
$6 \times 2 =$
$7 \times 2 =$
$8 \times 2 =$
$9 \times 2 =$
$10 \times 2 =$
$11 \times 2 =$
$12 \times 2 =$

Fig. 95

The advantages of the " Welbent " bead-bars will be seen when tables of numbers such as seven and six are being built up—see Fig. 96. The sections of complete fives when lying together show the tens plainly.

$7 \times 2 = 1 \quad 4$

Fig. 96

(3) **With Cardboard Strips.** Strips of cardboard 1″ wide, marked off in lengths of 1″ (so that each section forms a square), are useful for building up tables. They form a remote preparation for the later study of area. In schools where bead-bars cannot be provided these strips may be used instead. The method is shown in Fig. 97.

N.B. Loose beads are not good for building up tables of numbers above 5, not even when a perforated board is used.

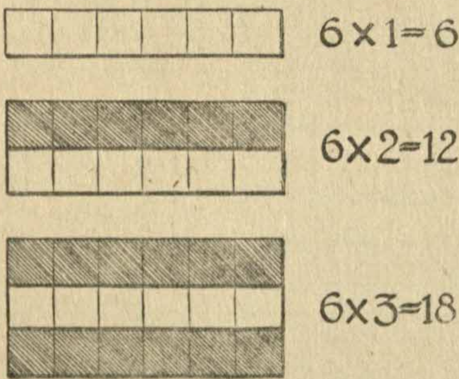


Fig. 97. Building up the Table of Sixes with Cardboard Strips

Each bead has to be picked up separately, and this makes the process unnecessarily slow, and does not encourage the child to calculate in groups.

(4) **With Number-chart.** Tables may be built up by putting counters on to a chart as in Fig. 98. For tables under 5 the child might at first be allowed to put a counter on each square, but after the first steps have been mastered counters ought to be put only on the squares giving the required products—*e.g.*, for the table of sixes counters

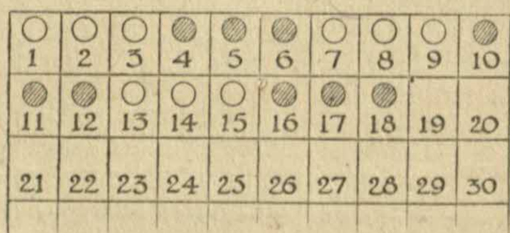


Fig. 98. Building up Table of Threes with Reversible Counters on Number-chart

would be put on the following squares only: 6, 12, 18, 24, 30, and so on. If each child is provided with eleven charts it is an excellent exercise for him to colour in one for each of the tables 2 to 12. These should then be kept for reference. On account of the patterns formed by each table these charts are an immense help in memorizing (see Figs. 99–110).

It is instructive for the child to build up the table of twos by putting reversible counters, black sides up, on all multiples

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 99. Table of Twos and Fours

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 100. Table of Threes and Sixes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 101. Table of Thirteens

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 102. Table of Nines

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 103. Table of Nineteens

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 104. Table of Sixes

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 105. Table of Eights

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 106. Table of Sevens

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 107. Table of Seventeens

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 108. Table of Elevens

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 109. Table of Twelves

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

Fig. 110. Table of Fives and Tens

of two, and then to build up the table of fours by reversing alternate counters. A similar exercise may be done for the tables of twos and eights, threes and sixes, threes and nines, and so on.

(The idea for these charts is taken from a little German book on numerical games which was published in the eighteen-nineties. Dr Montessori gives some charts in her *Advanced Montessori Method*, but apparently she uses them only for tables. For further uses see Chapter XII; also p. 188 of *The Psychology of Number*.)

Section IV. Memorizing Tables

On the Best Method of memorizing the Tables. The addition table has generally been left to look after itself, but the multiplication table has always received a certain amount of attention. In bygone days it was systematically memorized by frequent simultaneous repetition; and even at present the chanting of the tables, although less widely adopted, is almost the only means that is employed. But this chanting of the tables is open to several objections.

Memorizing of all kinds depends upon the fixation of a habit series; and the limits of the series should be clearly defined. Each formula, such as $4 \times 7 = 28$, constitutes a self-contained system, and it should be so memorized as to be completely usable without reference to preceding formulæ. In other words, no unnecessary associations should be set up. To associate by rote memory 4×5 with 4×6, and 4×6 with 4×7, etc., is a superfluous, if not an injurious, bit of mental mechanization. Chanting tends to establish these useless associations.

Another objection is that the speed of this simultaneous repetition is far too slow for the economic fixation of habit. The effect of speed upon mechanization, although not generally recognized, is considerable. If, for instance, a passage of poetry has to be memorized so as to render its repetition automatic, the repetition of the lines at maximum speed has been found, in my own case at least, to diminish the number of repetitions necessary. It has probably something to do with span of attention.

A third objection to the simultaneous chanting of tables is based upon the liability of the attention to wander during repetition. Attentive repetition is far more efficacious than the inattentive kind.

Finally, there is the objection that may be urged against all kinds of simultaneous class work; that is, that it makes no

allowance for individual differences in the mode and rate of learning.

There are experimental grounds for believing that the method of individual muttering—a method which unfortunately seems to 'get on the nerves' of some teachers—is considerably more efficacious than the method of concerted repetition.

There are two methods of memorizing the tables which I have reason to think would prove effective—methods which are not exclusive, but supplementary. *Both* might be tried.

(1) Take, say, two items per diem in the addition table, such as

X	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

Fig. 111. SUMMARY OF MULTIPLICATION TABLES

$7 + 8 = 15$, and $4 + 6 = 10$; and two per diem in the multiplication table, such as $7 \times 8 = 56$, and $4 \times 6 = 24$. If this be systematically done, and past work frequently revised, the whole will be learnt in less than five weeks. If only one of each be taken per day the whole can be mastered in less than three months.

(2) Learn by applying. Put, for example, the 7 times table before the boys and let them work *very rapidly* a large number of sums involving multiplication by 7. Of the two methods this is probably the better.[1]

Each child should have a set of tables for reference. These might be in any of the following forms:

(1) Fully stated as in Fig. 94.

(2) Picture-tables as in Figs. 99–110.

(3) Multiplication square as in Fig. 111.

[1] From *Mental Tests*, by Dr P. B. Ballard (Hodder and Stoughton).

This last is the most convenient form.

Constant practice should be given in writing out tables to time on cards as in Figs. 112 and 113. The latter is designed

X	1	2	3	4	5	6
2						
3						
4						
5						
6						

Fig. 112. CARD OF MULTIPLICATION TABLES UP TO 6 × 6

To be filled in by the child. "Progress" Games 2 and 3 (see p. 120 n.).

X	1	2	3	4	5	6	to 12
2	2	4					
4	4	8					
8	8	16					

Fig. 113. CARD TO SHOW CONNEXION BETWEEN THE TABLES OF TWOS, FOURS, AND EIGHTS

to bring out the connexion between the tables of twos, fours, and eights.

Fig. 114 shows a very useful form of drill-card. The exercise[1] consists in writing the product of each number

1	2	3	4	5	6	7	8	9	10	11	12
7	14	21	28	35	42	49	56	63	70	77	84
3	5	7	6	9	2	8	10	1	3	0	4
6	9	11	4	8	12	9	2	7	5	8	12
8	3	9	7	2	8	6	0	12	4	10	8
6	8	2	4	9	5	1	6	3	12	7	8

Fig. 114. DRILL-CARD ON THE TABLE OF SEVENS

Transparent paper is placed over the four rows of figures, and each product obtained by multiplying by 7 is written immediately over the number. The table of sevens is at the top for reference.

multiplied by seven. Transparent paper should be placed over the card, and the product written over the figures multiplied. If the child does not know a combination he should

[1] The idea for this exercise is taken from *The Courtis Standard Practice Tests*, by kind permission of Messrs Harrap.

refer to the table of sevens which is written at the top of the card. The answers should be printed on the back of the card so that the child can check his own results.

Another form of drill suggested by Dr Ballard in *Fundamental Arithmetic*, Part I, is to write numbers in any order

×	5	7	2	8	6	11	9	3	10	1	12	4
2												
3												
4												

Fig. 115. Drill-card on Tables

The answers should be on the back. Paper should be placed over the blank squares.

to form a top line and then to multiply them all by a series of numbers. Drill-cards might be prepared on these lines, a piece of paper for the answers being put over the blank spaces (see Fig. 115).

Section V. Exercises to introduce the Commutative Law and the Idea of Factors

The commutative law states that $a \times b = b \times a$. In other words, the factors may be interchanged without altering the product.

(1) Fig. 116 shows an exercise on the number 36. The square of Duplex counters may be formed on the table with all red sides up. To show 4×9 the child could either space the counters out in groups of four, thus ○ ○ ○ ○, or turn every alternate group over to the black side. The 9×4 would make four squares of 9, and the result would illustrate quarters well.

(2) Fig. 117 shows an exercise on the multiplication square (Fig. 111).

(3) "Progress" Game 3[1] provides cards and numerous

[1] This series of games, devised by the present author, provides practice in the fundamentals of arithmetic, prime numbers, factors, multiples, addition and subtraction, tables, money, and simple fractions. (Published by E. J. Arnold.)

exercises of a similar kind. A child could time himself in putting all the tickets on.

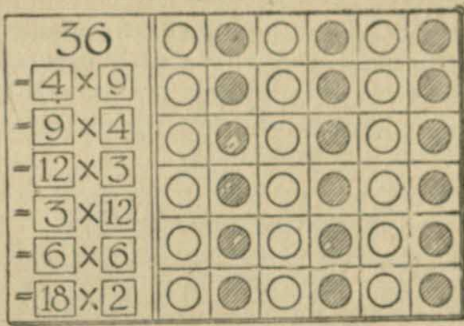


Fig. 116. Exercise to Emphasize Interchange of Factors

The 36 chart. Set out with Duplex counters to show that 36 is 6 taken 6 times. The child finds combinations of factors *in any order*, and then arranges similar factors together as shown in the result-card at the side. Similar cards may be used for numbers such as 12 ,18, 24.

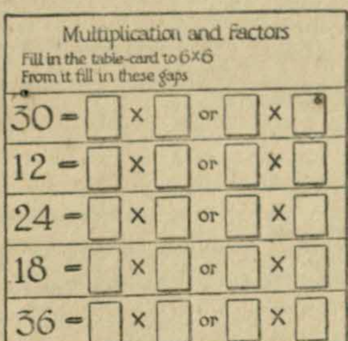


Fig. 117

The table-card is shown in Fig. 112.

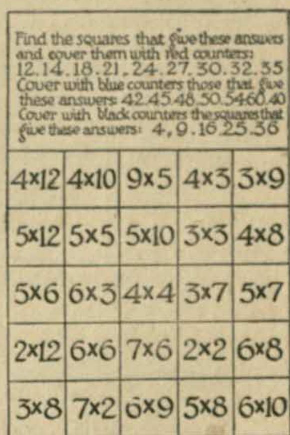


Fig. 118

(4) Fig. 118 shows another device for giving practice in tables. The card is self-corrective, as the pattern should be symmetrical or represent some object.

Section VI. Formal Written Work. Speed and Accuracy

Carefully constructed drill-cards are essential if speed and accuracy are to be acquired. The same drill-card may be worked through again and again, the answers only being written. To prevent the answers being memorized, the sums may be worked in a different order.

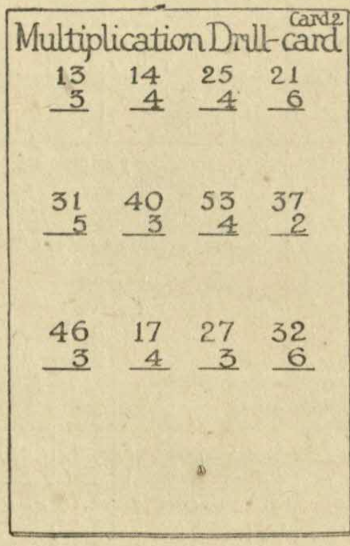


Fig. 119. Multiplication Drill-card

With sums spaced out so that it can be slipped into an exercise-book with the page cut into six equal strips.

Fig. 119 shows a specimen card for small children working sums as in (2) on p. 124. The advantage of this type of card lies in the ease with which strips of paper can be laid along the bottom of each row for the answers. If the pages of an ordinary exercise-book are cut into strips, the card can be put into the book with a strip laid in place for the answers (see Fig. 120).

It is essential that the child be taught to say or think only the necessary words. For instance, in the example 36 × 4 he should say or think only "24," and write 4, then "12,

14," and write 14. He must not say " 6 taken four times " or " 4 times 6 equals 24. Put down 4 and carry 2. 4 times 3 are 12. Add in 2. That makes 14. Put it down ! " All this rigmarole interferes with rapid work and clear thinking.

The very best practice that can be given in tables is by

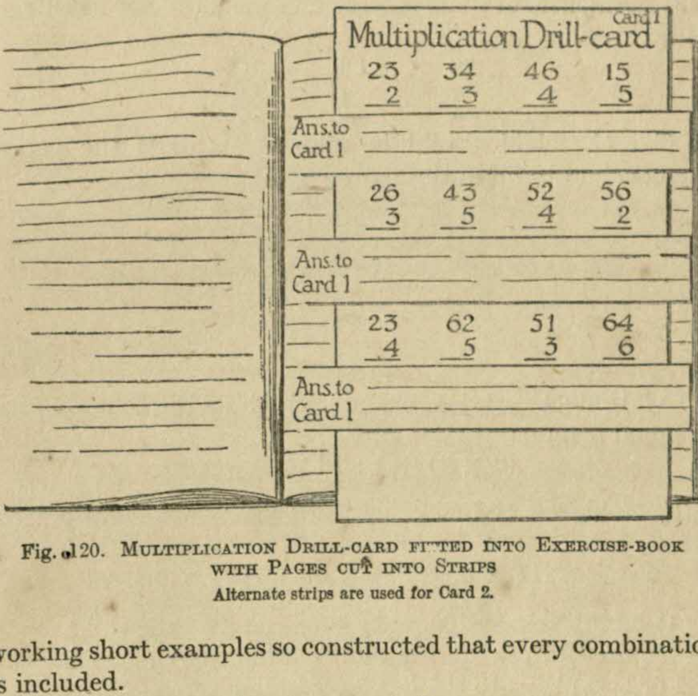


Fig. 120. MULTIPLICATION DRILL-CARD FITTED INTO EXERCISE-BOOK WITH PAGES CUT INTO STRIPS

Alternate strips are used for Card 2.

working short examples so constructed that every combination is included.

The following grading is suggested :

(1) **Multipliers of Units only, Multiplicand of Tens and Units.** No carrying—*e.g.*,

$$23 \times 2. \qquad 43 \times 2. \qquad 23 \times 3.$$

Here, as in future and more difficult examples, it is a good plan to let the child write down partial products. He should be allowed to suggest ways of working, so that he realizes fully that 23×2 is really $(20 \times 2) + (3 \times 2)$.

This is, of course, an application of the distributive law, which states that if a number x be multiplied by the sum of

several numbers $(a + b + c)$ the product is the same as the sum of the products of each number multiplied by x.

I.e., $x(a + b + c) = ax + bx + cx$.

E.g., $6\,(562) = (500 \times 6) + (60 \times 6) + (2 \times 6)$.

(2) **Multipliers of Units only, Multiplicand of Tens and Units.** With carrying—*e.g.*,

23	32	34
4	4	4

At first the child should be allowed to write the partial products and to fill in the zeros—*e.g.*,

$$\begin{array}{r} 23 \\ 4 \\ \hline 12 \\ 80 \\ \hline 92 \end{array}$$

To introduce carrying, examples such as the following will be found helpful:

$$(3 \times 4) + 1\,;\ (6 \times 3) + 2.$$

(3) **Multipliers of Units only, Multiplicands of Three or Four Places.** *E.g.*,

$$214 \times 3. \quad 524 \times 4. \quad 1024 \times 5.$$

(4) **Multiplying by 10 and by 100.** At first the child should be required to rewrite the multiplicand, moving each figure one or two places to the left, and adding zeros. A few exercises with digit-tablets on paper ruled in columns would be a help.

(5) **Multiplying by 20, 30, 40, etc.** At first the child will multiply by 10, then by 2.

$$\begin{array}{r} 34 \\ 20 \\ \hline 340 \\ \hline 680 \end{array}$$

Then he must be shown how to multiply by 10 and 2 simultaneously. This will require some practice.

(6) **Multiplying by any Number under 100.** If the preceding stages have been grasped this should cause no difficulty.

The left-hand figure of the multiplier should be used first, thus:

```
  324        Not   324
   36               36
 ----             ----
 972              1944
 1944              972
 -----            -----
 11664            11664
```

The advantages of this are that by it the main part of the answer (in this case 9720 out of 11664) is obtained at once and that later on when approximations are taught this method is necessary. It is no more difficult than the other.

Section VII. Short Methods. Factor Multiplication

Short methods are extremely useful in multiplication, but it is a mistake to teach them before ordinary long multiplication. All short methods should come as a result of working the long methods. A child who has multiplied 3786 by 99 will appreciate the 'trick' (which he should be required to find out for himself) of multiplying by 100 and then subtracting 3786.

A good way of inducing children to find out short methods without emphasizing them prematurely is to require the sums to be checked. Multiplication sums may, of course, be checked by division, but this is sometimes unnecessarily long. It might, therefore, be suggested to the child to find another way of working the example, and this would lead to experiments in short methods.

Multiplication by factors is not of universal application, and even when the multiplier is easily factorized the working is not always simplified thereby. For instance, it is quite as quick to multiply 79 by 28 as to multiply it first by 7 and then the product by 4. It seems far the best plan to teach children under eleven to work long multiplication rapidly, and then later to introduce factor multiplication for special cases.

CHAPTER VII

DIVISION

SECTION I. THE NATURE OF DIVISION: PARTITION AND QUOTITION

WE can look upon the expression 12 ÷ 3 from two points of view. First, we might read it as "Twelve divided by three," or "Twelve grouped into threes"; this would give us as answer the number of groups—viz., 4. Secondly, we might read it as "Twelve divided into three groups," and this would also give us 4, but would tell us the number in each group.

It is very important that teachers should understand this twofold nature of division. On it depends first of all the manipulations which a child will make with concrete material; secondly, the nature of the answer, whether it be concrete or abstract; and thirdly, the nature of the remainders.

These two aspects of division are known as partition and quotition (from Latin *quoties*, 'how many').

(1) **Partition: Sharing, dividing into, between, or among.**

Examples:

> Divide 28 into 4 equal parts. How many in each part
> What is one quarter of 28 ?
> Five yards of cloth cost 42*s*. 6*d*. How much does 1 yard cost ?
> If 12 apples are shared among 3 boys, how many will each get ?

In working this last example the child would take 12 counters and deal them out, giving the boys first one each, then a second one each, and so on. The result would be arranged as follows:

A's	B's	C's
O O O O	O O O O	O O O O

In this case the answer is a concrete quantity—viz., "4 apples." In all examples of partition we are given the number of groups, and are required to find their size—*i.e.*, of how many or how much each group consists.

Partition is really fractional division, for when we say "Divide 12 among 3 boys" we mean "Give a third of 12 to each boy."

(2) **Quotition: Measuring, grouping, dividing by.**

Examples:

How many times can I subtract 5 from 75?
How many fives must I take to make 75?
75 divided by 5.
What part of 75 is 5?
How many pence in 252 farthings?
I have 12 apples. To how many boys can I give three?

In working this last example the child would again take 12 counters, but this time, as he knows the size of the groups, he would lay the counters out in groups of three, thus:

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

The answer would be abstract—4 (*i.e.*, to 4 boys). We may regard quotition as measuring up one quantity by another quantity, or as continued subtraction.

We have seen that with concrete apparatus the expression $12 \div 3$ may be represented either as 3 groups of 4,

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

o as 4 groups of 3,

○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○ ○

The first arrangement corresponds to 4×3, or "four taken three times," and the second arrangement corresponds to 3×4, or "three taken four times." Thus the two aspects of division are connected with the principle of commutation of factors in multiplication. Whether we measure up the 12 units in threes or share them out into 3 groups we get the same *number* in our answer. What that number stands for must be decided by the nature of the problem.

(3) **Concrete and Abstract Quantities.** Thorndike says in his

Psychology of Arithmetic (Macmillan), p. 85 (under the heading of "Wasteful and Harmful Bonds"):

> *Abstract and Concrete Numbers.* The elaborate emphasis of the supposed fact that we cannot multiply 726 by 8 dollars and the still more elaborate explanations of why nevertheless we find the cost of 726 articles at $8 by multiplying 726 by 8 and calling the answer dollars are wasteful. The same holds of the corresponding pedantry about division. These imaginary difficulties should not be raised at all. The pupil should not think of multiplying men or dollars, but simply of the necessary equation and of the sort of thing that the missing number represents. "$8 \times 726 = \ldots$ Answer is dollars," or "8, 726, multiply . . . Answer is dollars," is all that he needs to think, and is in the best form for his thought.

When difficulties of this sort are explained to children they are apt to think the answer to any division sum a 'tricky' thing, and the result of this uncertainty is that they sometimes resort to guessing. *If the nature of the answer is treated from the beginning as a matter for common sense children will experience little difficulty in fitting answer to question.*

"How many pennies are there in 252 farthings?" requires as answer 63.

"How many times does 4 go into 252?" *Answer*, 63 (times).

"What part of 63 is 9?" *Answer*, one-seventh.

"5 yards cost 42*s.* 6*d.* How much does one yard cost?" *Answer*, 8*s.* 6*d.*

"What is a quarter of 28?" *Answer*, 7.

(4) **Remainders.** There are two ways of expressing a remainder. For instance, in the example $247 \div 7$ we may express it either as in (*a*) or (*b*):

(*a*) 7)247		(*b*) 7)247
35 rem. 2		$35\frac{2}{7}$
or 35 r. 2		

Which expression we use will depend on the stage of development of the children, and on the question being dealt with. For children under eight statement (*a*) is best, but after that age they should be able to use the fractional method *when* the fractions are simple ones such as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$. The explanation should be simply that $\frac{1}{2}$ represents "one divided by two."

If the child has been introduced to fractions or even to the fractional form for farthings and halfpennies he will not find this too difficult.

It happens sometimes in problems that the fractional representation is not suitable.

"How many complete cricket elevens can be formed with 25 boys?" Obviously only 2. If only 25 boys were available, 3 would have to fall out. Likewise answers which give a fraction of a boy are not for the junior school! All problems given to small children should be on practical, everyday occurrences. A problem such as the following was given to a little girl of seven and a half: "A basket holds 8 oranges. How many baskets can I fill if I have 50 oranges?" She gave as answer, "6 baskets and 2 oranges left over." It was pointed out to her that "the book" said "6¼ baskets." Whereupon she responded, "But a ¼ basket is not a full basket; it's only ¼ full, not a basket full!" Which answer was very sensible.

Unfortunately, in some schools the following method of expressing remainders is allowed:

$$7\overline{)247}$$
$$\overline{35 + 2}$$

This, of course, is an absurdity, for 35 + 2 equals 37. Here the fact that the 35 and the 2 are of different denominations has been ignored. This method has in some cases been found to arise from the plus sign being interpreted as "and." The form + 2 is then taken as an abbreviation of "and two left over." This fault is comparable to the misuse of the equals sign in "Answer = 35 r. 2"!

Section II. Beginnings in Division

In spite of the fact that partition or sharing is really fractional division, and that hence some authorities maintain that it should not be taught until fractions are dealt with, it remains certain that the idea of sharing as dealing out is more familiar to a child than that of measuring. Constantly in real life objects have to be shared among people. Hence

psychologically it seems sound to let a small child begin division by sharing. If Johnny has 6 apples to share with his brother and sister, the important point to him is that he fets 2 apples. That two is one-third of six will not, and need got, trouble him. Later, when dealing with fractions, this ractional aspect may be studied, and these early experiences will help to make it clear.

As soon as a child has had practice in 'sharing' sums with numbers up to 20 or 30 he should be introduced to grouping by such an example as "How many threes are there in 15 ?" If the work is carefully arranged there is no danger of confusion, for the question decides the manipulation. When the method is not specified as sharing, grouping should be used. If a child is left to himself, as he should be, to experiment in sharing out counters and measuring up one quantity by another, he is bound to come across examples with remainders. The usual practice of avoiding examples with remainders and of postponing them until later is a great mistake. At first it will be sufficient if the child registers them as in the following example:

$$13 \div 3 = 4 \text{ r. } 1.$$

This has already been discussed in Section I.

The apparatus recommended for division is the same as that used for the other rules:

(1) Duplex counters—for very simple sharing sums.
(2) Long bead-bar—for very simple grouping sums.
(3) Short bead-bars for sharing with higher numbers and for teaching short division.
(4) Bone counters in at least four colours for short and long division.

Section III. First Exercises in Division

(1) **Sharing.** *With Counters.* After some experimenting and building up problems and equations in sharing the child should be given sum-cards of the type shown in Fig. 121. In the first example the child takes 9 counters, and deals them out by placing them on the pictures—"One to Don, one to Dan," and so on until no counters are left.

When this process is understood the child should be taught to place his counters vertically under one another, as in Fig. 122. The names of the dogs are written on slips of paper or card, and each one's counters are put beside its name. This arrangement is *very* important, as will be seen when short division is dealt with. Every division sum, whether in sharing or grouping, should be set out in vertical lines, as much confusion is saved in this way and the work becomes clear and orderly.

Share among Don, Dan, and Jip

9 buns	each
12 tit-bits	each
6 rats	each
15 crusts	each
18 bones	each

Fig. 121. First Type of 'Sharing' Card (with Pictures)
"Welbent" Series, Stage 4, Step 10.

Don's share	3 buns
Dan's share	3 buns
Jip's share	3 buns

Fig. 122. Showing Method of Work—"One for Don, one for Dan"

Fig. 123 gives another type of card. The method of working is shown in Fig. 124. Examples with remainders appear on the card in Fig. 125. Here the cakes are not to be cut; hence the number of whole cakes over is registered under "remainder." Some cards have no examples with remainders. This is done purposely to accentuate the relation to the multiplication tables. This relation of division to multiplication the child

Sharing apples among Mary, John, Jim, and Pat

apples	apples
$12 \div 4 =$	each
$8 \div 4 =$	each
$4 \div 4 =$	each
$16 \div 4 =$	each
$20 \div 4 =$	each
$24 \div 4 =$	each

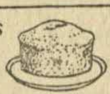
Fig. 123. Second Type of 'Sharing' Card (no Remainders)

Mary's share	3 apples
John's share	3 apples
Jim's share	3 apples
Pat's share	3 apples

Fig. 124

should discover for himself. It often comes to him as a revelation, and the joy of discovery is taken from him if the teacher points the connexion out. Only after it has been

Sharing cakes among 3 boys



Cakes	Cakes Rem.
3 ÷ 3 =	
9 ÷ 3 =	
7 ÷ 3 =	
10 ÷ 3 =	
12 ÷ 3 =	
14 ÷ 3 =	

Fig. 125. Third Type of 'Sharing' Card (with Remainder)

0 0 0 0
0 0 0 0
0 0 0 0 12 eggs

How many groups		
of 3		12 ÷ 3 =
of 4		12 ÷ 4 =
of 2		12 ÷ 2 =
of 6		12 ÷ 6 =
of 12		12 ÷ 12 =

Fig. 126. First Type of 'Grouping' Card

discovered and fully investigated should exercise-cards as in Fig. 133 be given.

(2) **Measuring or Grouping.** (*a*) *With Counters.* (i) Fig. 126 shows an exercise-card on the number 12. Similar exercises may be devised by arranging counters in rectangles.

Grouping 24 Marbles
24 ÷ 2 =
24 ÷ 3 =
24 ÷ 4 =
24 ÷ 6 =
24 ÷ 8 =
24 ÷ 12 =

Fig. 127. Second Type

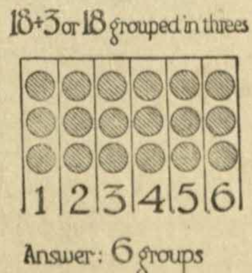


Fig. 128. Showing Method of Working

Fig. 127 might be worked through with twenty-four counters.

Similar cards might be prepared dealing with numbers such as 48, 36, 40, etc.

(ii) The most satisfactory method of setting out the counters is shown in Fig. 128. Here the counters are arranged in groups vertically, so that by counting the columns the quotient is obtained. The arrangement should be compared with that shown in Figs. 122 and 124 for sharing.

(*b*) *With the Long Bead-bar.* A number of beads—let us say 20—is marked off by a clip or tag. The child then sets to work to measure off the 20 beads in twos, threes, and so on. Cards like those in Figs. 129, 130, and 131 are useful.

Grouping 20 Sweets	
20 ÷ 10 =	Rem.
20 ÷ 9 =	
20 ÷ 8 =	
20 ÷ 7 =	
20 ÷ 6 =	
20 ÷ 5 =	
20 ÷ 4 =	
20 ÷ 3 =	
20 ÷ 2 =	
20 ÷ 1 =	

Fig. 129

Grouping in Threes
6 ÷ 3 =
12 ÷ 3 =
24 ÷ 3 =
9 ÷ 3 =
18 ÷ 3 =
36 ÷ 3 =

Fig. 130

Some teachers prefer that in Fig. 132. Here the divisors are given, and the child is obliged to take them in order. This has the grave disadvantage of placing one of the most difficult examples first—viz., 10 ÷ 1. This should really come after the child has worked 10 ÷ 3 and 10 ÷ 2, for grouping or measuring in ones is difficult to understand.

(*c*) *With Number-rods.* Children will find these exercises more difficult than those with the long bead-bar. Starting with the 5-rod placed on the top of the 10-rod, the child marks with his finger the point which the 5-rod reaches on the 10-rod, and then moves the 5-rod up, saying, " One, two." This result is registered as " 10 ÷ 5 = 2."

The 10-rod has been measured and its value found in terms of the 5-rod.

The 10-rod is then measured with the 4-, 3-, 2-, and 1-rods.

Though the graduation marks on the rods help to fix the points, marking with the finger is not sufficiently accurate. It is better for the child to use a piece of wire bent into two right angles so as to fit the 10-rod.

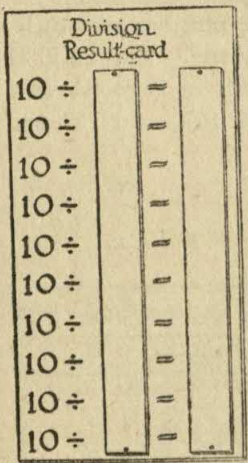


Fig. 131. Result-card, with Strips of Paper fixed on for Divisors and Quotients

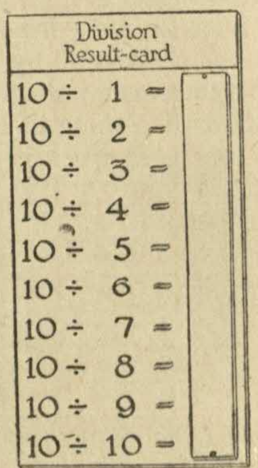


Fig. 132. Result-card, with Strip of Paper for Quotients only

Section IV. Introduction to Short Division

(1) Multiplication and Division Tables. As soon as a child has realized the connexion between his multiplication tables and simple division sums he might be given a card like that in Fig. 133. This he could work through orally. The multiplication square (Fig. 111) should also be used for breaking up numbers into their factors. For instance, the child might be required to find the factors of 24. This number occurs six times on the chart, once in connexion with each of the following pairs of factors: 2×12; 12×2; 6×4; 4×6; 8×3; 3×8.

When this and similar exercises have been worked, the following form for writing down a division sum should be taught:

$$\begin{array}{r} 6\overline{)24} \\ \hline 4 \end{array}$$

At this stage the child will enjoy converting his multiplication tables into division tables.

(2) **Exercises with the Short Bead-bars.** These exercises are given for the sake of completeness, and because there will

Multiplication	Division
9 × 2 =	18 ÷ 2 =
6 × 3 =	18 ÷ 3 =
2 × 9 =	18 ÷ 9 =
8 × 3 =	24 ÷ 3 =
4 × 6 =	÷ 6 = 4
12 × 2 =	÷ 2 = 12

Fig. 133. EXERCISE TO SHOW THE RELATION BETWEEN MULTIPLICATION AND DIVISION

" Welbent " Series, Stage 5, Step 1.

always be found some teachers who prefer them to those described in Section V, which are worked with coloured counters. The exercises with short bead-bars are, of course, more elementary, for 100 is actually represented by 100 beads.

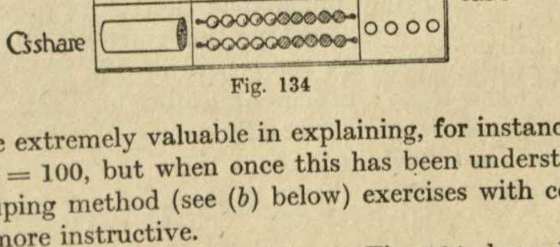


Fig. 134

They are extremely valuable in explaining, for instance, that $200 \div 2 = 100$, but when once this has been understood by the grouping method (see (*b*) below) exercises with counters are far more instructive.

(*a*) *Sharing with Short Bead-bars.* Fig. 134 shows how an example of the following type might be worked: " 372 marbles are to be divided among three boys. How many will each get ? "

The process for all such examples is as follows :

Share out the hundreds.
Change any remaining hundreds to tens.
Share out the tens.
Change any remaining tens to units.
Share out the units.

(*b*) *Measuring or Grouping with Short Bead-bars.* Fig. 135 shows the following example worked : " How many threes are there in 372 ? "

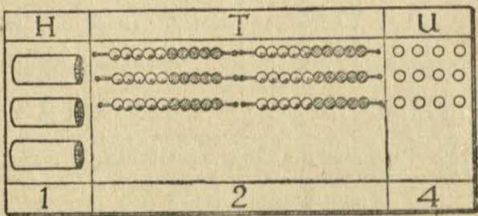


Fig. 135

The process for all such examples is as follows :

Group the hundreds, and record number of complete groups.
Change any remaining hundreds to tens.
Group the tens, and record number of complete groups.
Change any remaining tens to units.
Group the units, and record number of complete groups.

Though a child may understand that there are 2 groups of 3 in 6, he may not understand that there are 20 groups of 3 in 60, or that there are 100 groups of 3 in 300. It is for explaining this that bead-bars are so useful. The child can put bead-bars out and actually count the groups. For instance, to see how many twos there are in 40 he would place the bead-bars as shown in the third row of Fig. 136. For $300 \div 3$ three 100-chains could be laid alongside one another, and groups of 3 beads (one on each chain) counted off. Long before reaching the 100 the child will see that there are 100 threes in 300. This is very important, for unless the child has mastered it he will constantly get confused with

regard to the place-value of his quotients. *However, too much time should not be spent explaining. The point is best grasped by actually working examples.*

Figs. 137 and 138 are examples of very useful types of drill-cards. They are designed to emphasize place-value.

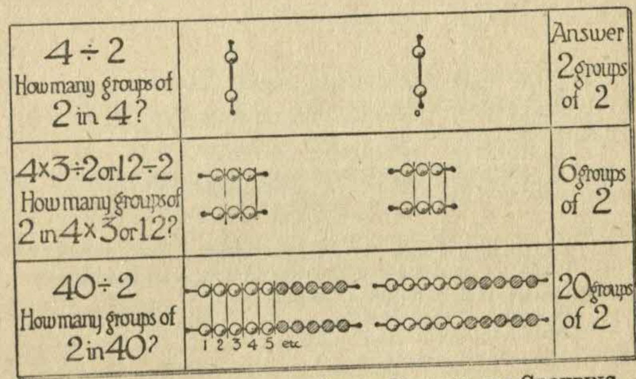


Fig. 136. Explanation of Short Division by Grouping

Grouping in Threes
3 ÷ 3 =
30 ÷ 3 =
36 ÷ 3 =
6 ÷ 3 =
60 ÷ 3 =
69 ÷ 3 =
9 ÷ 3 =
90 ÷ 3 =
93 ÷ 3 =

Division by Grouping
4 ÷ 2 =
40 ÷ 2 =
400 ÷ 2 =
6 ÷ 2 =
60 ÷ 2 =
600 ÷ 2 =
8 ÷ 2 =
80 ÷ 2 =
800 ÷ 2 =

Figs. 137, 138. Introductory Exercises to Short Division

Section V. Short and Long Division

(1) **Short Division with Apparatus.** Figs. 139 and 140 show what seems the best way of working a division sum with apparatus. Here the value of each counter must be understood by the child. No special boards are needed; trays

and tablets marked "Th," "H," "T," "U," are all that are required. The use of counters has an advantage over an abacus, because all the working can be seen, whereas with

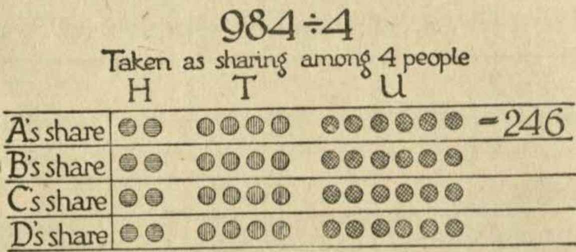


Fig. 139

Deal out the hundreds. There are 2 each, and one over.
Change the remaining one for 10 tens. Now there are 18 tens.
Deal them out. There are 4 each, and 2 over.
Change the 2 tens for 20 units.
Deal out the units. There are 6 each.
∴ Each person gets 246.

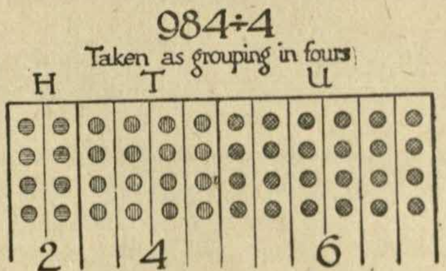


Fig. 140

Group the hundreds in fours. There are 2 groups and one 100-counter over. That is, 4 goes 200 times into 800.

Change the 100-counter into 10 10-counters. There are 18 tens now. Group these 18 counters into fours. There are 4 groups and 2 counters over. That is, 4 goes 4 times into 16.

Change the 2 10-counters for 20 unit counters. There are now 24 unit counters. Group them in fours. They form 6 groups.

Answer: 246 groups of 4 in 984.

an abacus only the answer is left at the end, so that if a mistake has occurred it is difficult to trace.

There is no necessity for very long examples to be given. This apparatus is intended only to teach the process. After an example has been worked with apparatus it should be written down and worked without apparatus. An occasional

return to apparatus is valuable as a revision of the fundamentals underlying the process, but to work all examples with apparatus, when once the process is understood, is a waste of time.

(2) **Long Division.** The same apparatus may be used as for short division. A description of the process is given below in (3).

Long division is the most difficult process met with in the four rules, yet it is indispensable. Sooner or later it must be learnt, and time is better spent on it than on division by factors, where the remainders are most confusing and the method not of universal application.

Some authorities advocate postponing it as long as possible (see Thomson, p. 74, Potter, p. 57). Yet its universal applicability makes the knowledge of it essential. The fault in the past seems to have been in giving children examples which were *too difficult* and in having no apparatus to explain the process. In some schools excellent results have been obtained by teaching long division first and short division later. In this way the child from the beginning realizes the form of the sum and gets into the habit of writing all his figures down. For example, the sum worked with apparatus in Figs. 139 and 140 would appear as in (*a*) or (*b*), instead of as in (*c*).

(*a*)
$$\begin{array}{r} 246 \\ 4\overline{)984} \\ \underline{800} \\ 184 \\ \underline{160} \\ 24 \\ 24 \end{array}$$

(*b*)
$$\begin{array}{r} \underline{246} \\ 4)984 \\ \underline{8} \\ 18 \\ \underline{16} \\ 24 \\ 24 \end{array}$$

(*c*)
$$\begin{array}{r} 4)\underline{984} \\ 246 \end{array}$$

In (*a*) zeros are added, as some teachers feel that this makes the child realize the place-value of the quotient. This habit of adding zeros must be given up later, however.

The grading of long division sums is carefully worked out in *Fundamental Arithmetic*, by Dr Ballard. The first examples should be such as require only a good knowledge of the

multiplication tables, followed then by those in which the figures of the quotient are found by considering only the first figure of the divisor—*e.g.*, 253 ÷ 23. A good deal of practice should be given in examples that have zeros in the quotient or dividend—*e.g.*, 4669 ÷ 23 = 203. Children should be taught after every subtraction to look and make sure that the product is not larger than the partial product and that the remainder is smaller than the divisor. Checking by multiplying quotient by divisor or by dividing the dividend by the quotient should be encouraged.

Original sum :	284 (Dividend)	÷	4 (Divisor)	=	71 (Quotient)
Check 1 :	284 (Dividend)	=	4 (Divisor)	×	71 (Quotient)
Check 2 :	284 (Dividend)	÷	71 (Quotient)	=	4 (Divisor)

(3) **Long Division with Apparatus.** Here we have to choose between Dr Montessori's elaborate set of stands, tubes, beads, and boards and the simpler method with coloured counters. The latter has the advantage of being cheaper and less cumbersome, and more durable. The former is, however, very perfect, and with it any example, no matter how complicated, may be worked. The tubes and boards are so fascinating that even grown-ups often cannot tear themselves away, and are seen engrossed in working such an example as 3,560,895 ÷ 37. However, for ordinary school purposes it is not desirable to work such long examples with apparatus.

(*a*) *Long Division with Counters* (see Fig. 141).

Example : 2852 ÷ 23.

Put out 2 orange, 8 green, 5 red, and 2 yellow counters to represent thousands, hundreds, tens, and units respectively.

Arrange the orange counters, which represent thousands, in twos. There will be 1 group.

Arrange the green counters, which represent hundreds, in threes. There will be 2 groups and 2 over. Of these 1 group only is used, for there is only 1 group of 2 in the thousands.

Record 1 with a digit-tablet in the hundreds place. That is, 23 goes 100 times into 2800, and there are 500 over.

Arrange the 5 remaining green counters in groups of two. There will be 2 groups of two and 100 over. Change this green counter for 10 red ones.

Arrange the 15 red counters, which represent tens, in groups of three. There will be 5 groups. Of these two only are needed. 9 remain.

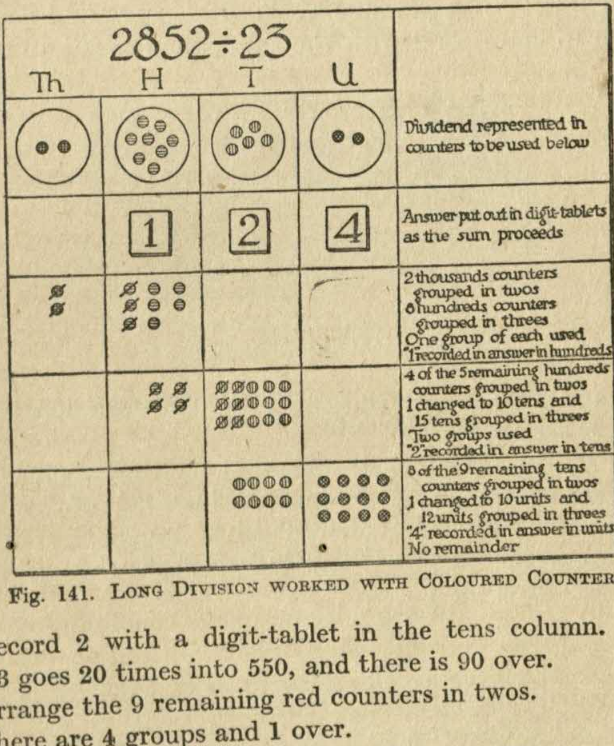


Fig. 141. Long Division worked with Coloured Counters

Record 2 with a digit-tablet in the tens column. That is, 23 goes 20 times into 550, and there is 90 over.

Arrange the 9 remaining red counters in twos.

There are 4 groups and 1 over.

Change the remaining 1 for 10 yellow counters.

Arrange the 12 yellow counters in groups of three. There will be 4 groups.

As there are 4 groups of two in the tens, and 4 groups of three in the units, record 4 with a digit-tablet in the units column.

The answer, therefore, is 124.

(*b*) *Montessori Long Division Apparatus.* This consists of seven stands each holding ten glass tubes. Each tube

contains ten beads coloured as on an abacus to indicate place-value (see Fig. 142).

The stands also are coloured—one shade for units, tens, and hundreds stands, a second shade for the next three stands—viz., thousands, tens of thousands, and hundreds of thousands—and a third shade for millions.

There are also seven round wooden dishes or boxes coloured outside to correspond with the stands and inside to match

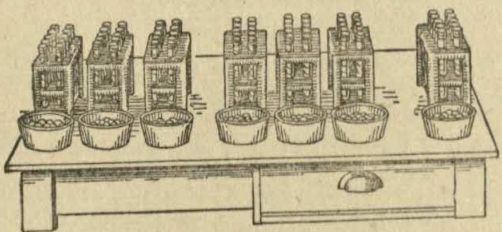


Fig. 142. MONTESSORI LONG DIVISION APPARATUS
By permission of Messrs Philip and Tacey, Ltd.

the beads. Two or three perforated multiplication boards complete the apparatus.

Two (if the divisor is two-figured) multiplication boards are placed in front of the millions and hundreds of thousands stands, each of which has its own dish of beads.

The working of a long example will be found in *The Advanced Montessori Method.*

SECTION VI. SPEED AND ACCURACY

A thorough knowledge of the multiplication tables is the first essential if rapid and accurate work is to be done.

The exercises must be carefully graded, and any number of simple examples should be worked for the sake of cultivating speed.

Practice should be given in finding trial divisors. For example, when 19 is the divisor the children should realize that, 19 being so near twenty, 2 and not 1 should be taken as trial divisor. Drill such as the following will be found helpful:

294 divided by 2, 20, 21, 19, 22.
966 „ „ 2, 4, 20, 40, 19, 21, 42.
1238 „ „ 3, 30, 15, 20, 25, 31.

A number-chart from 1 to 200 is invaluable for drill in division (see Chapter XII).

A multiplication chart extending to the nineteen times table would be useful for the child to refer to. The children

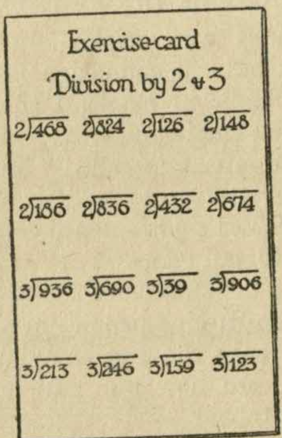


Fig. 143

should know the relation of 20, 25, 13, etc., to 100, and should be able to make out these tables rapidly.

Drill-cards like that shown in Fig. 143 should be used (*cf.* Fig. 119). In the card shown the quotient is to be written above the divisor, even though short division is used. The advantage of this is that as the answer has to be placed thus in long division sums it is well to get the child accustomed to it. However, many teachers hold, and with reason, that to write the quotient below the dividend is quicker and more natural.

What has been said with regard to short methods in multiplication applies also to division (see Chapter VI, Section VII).

CHAPTER VIII

MONEY CALCULATIONS

Section I. History of the British Money Table

In olden times money was weighed, not counted; hence there came to be a natural connexion between weight and money value. In medieval England the actual weight of 240 silver pennies was the pound troy, so it came about that the gold coin of that value was called "a pound." The sign £ comes from the Roman word *libra*, "a balance."

The *d.* which we use to signify pence comes from *denarius*, one of the earliest Roman coins.

The *s.* in our "£ *s. d.*" stands for the Anglo-Saxon *scilling*. The fact that we have twelve pennies in a shilling is said to be due to the preference the Scandinavians showed for the number 12. So we see how our national history can be traced in our money table!

For an interesting account of our money system students are referred to *The Story of Arithmetic*, by Susan Cunnington (Allen and Unwin).

Section II. First Steps in Money Calculations

There are two chief difficulties met with in teaching small children to work money sums.

(1) The children are apt to confuse tens and twelves when carrying or reducing.

(2) They very often do not know the multiplication table of twelves, and are too young to learn it.

The use of the money calculation cards shown in Figs. 144 and 145 (the "Riverside" Money Charts, published by Philip and Tacey) is a great help in overcoming these

difficulties, for they give a good visual representation of the essential facts to be learnt—viz., that

4 farthings make 1 penny.
2 halfpennies make 1 penny.
12 pennies make 1 shilling.
20 shillings make 1 pound.

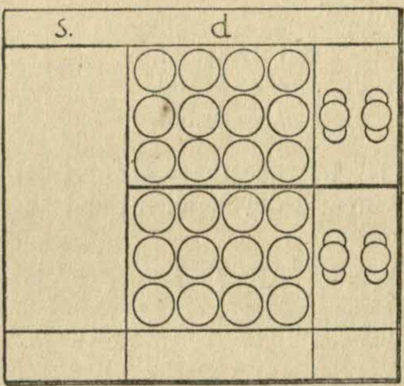


Fig. 144

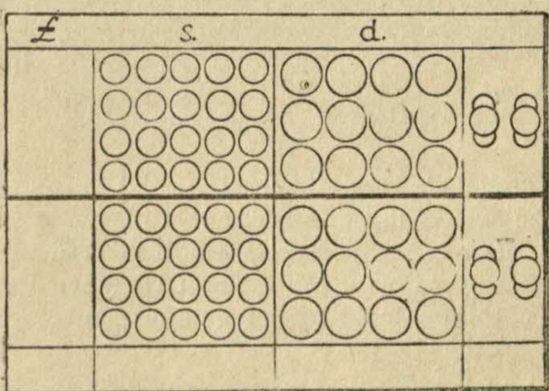


Fig. 145

They enable a small child who does not know his table of twelves to work simple sums by completing twelves and changing each twelve into a shilling immediately. This process, of course, entails only an analysis of 12.

The first difficulty is also avoided by using completely different apparatus from that employed in decimal notation; hence no beads or sticks should be used. Coins are essential, and there should be a very liberal supply of them.

Some teachers attempt to overcome the second difficulty by making the children learn the pence table. The following is a very common form of it:

$$20d. = 1s.\ 8d.$$
$$30d. = 2s.\ 6d.$$
$$40d. = 3s.\ 4d.$$

And so on. This is a mistake, for to memorize without understanding retards mental development, and the introduction

	1	2	3	4	5	6	7	8	9	10	11	12	1s.
1s.	13	14	15	16	17	18	19	20	21	22	23	24	2s.
2s.	25	26	27	28	29	30	31	32	33	34	35	36	3s.
3s.	37	38	39	40	41	42	43	44	45	46	47	48	4s.
4s	49	50	51	52	53	54	55	56	57	58	59	60	5s.
5s	61	62	63	64	65	66	67	68	69	70	71	72	6s.

Fig 146. NOT RECOMMENDED

of the 20*d.*, 30*d.*, 40*d.*, very often leads a child to confuse the tens with the twelves in the pence.

It seems better to postpone mechanical memorizing until the child is ready to learn the table of twelves; then the building up of this pence table forms a most useful exercise.

Neither does the chart shown in Fig. 146 seem altogether satisfactory, for it is constructed for the money tables only, and cannot be used for any other purpose—it shows no other facts clearly except the table of sixes and certain multiples of 3. A chart to the basis of ten, on the other hand, is of very wide application, and for money sums such a chart could have the multiples of 12 coloured or framed so as to stand out and show clearly the interesting symmetry of their

arrangement (see Fig. 147). If a child is allowed to refer to such a chart he will soon learn his table of twelves. (For small children a chart to 60 is sufficient, but later it should

1	2	3	4	5	6	7	8	9	10
11	$\boxed{12}$	13	14	15	16	17	18	19	20
21	22	23	$\boxed{24}$	25	26	27	28	29	30
31	32	33	34	35	$\boxed{36}$	37	38	39	40
41	42	43	44	45	46	47	$\boxed{48}$	49	50
51	52	53	54	55	56	57	58	59	$\boxed{60}$
61	62	63	64	65	66	67	68	69	70
71	$\boxed{72}$	73	74	75	76	77	78	79	80
81	82	83	$\boxed{84}$	85	86	87	88	89	90
91	92	93	94	95	$\boxed{96}$	97	98	99	100

Fig. 147. Recommended as of Very Wide Application

be extended to 150 if possible. For other uses of charts see Chapter XII.)

To sum up, therefore, the two main difficulties may be overcome

(1) By using cardboard coins and not apparatus similar to that used in decimal notation.
(2) By providing cards which give good visual representation of the underlying facts.
(3) By allowing the children at first to build up each set of twelve pence, which does not imply knowledge of tables.
(4) By having a chart to the basis of ten to which the child can refer, and on which he can build up tables.

The exercises suggested for the teaching of money calculations may be grouped under the following headings:

(1) Recognition and relative value of coins.
(2) Simple exercises involving the four rules.
(3) Compound addition.
(4) Compound subtraction.

(5) Shopping activities.
(6) Compound multiplication.
(7) Compound division.

Section III. Recognition and Relative Value of Coins (all Exercises to be worked with Coins)

(1) **Recognition of Coins.** Much time need not be spent on this, as the results of not knowing the value of coins are in

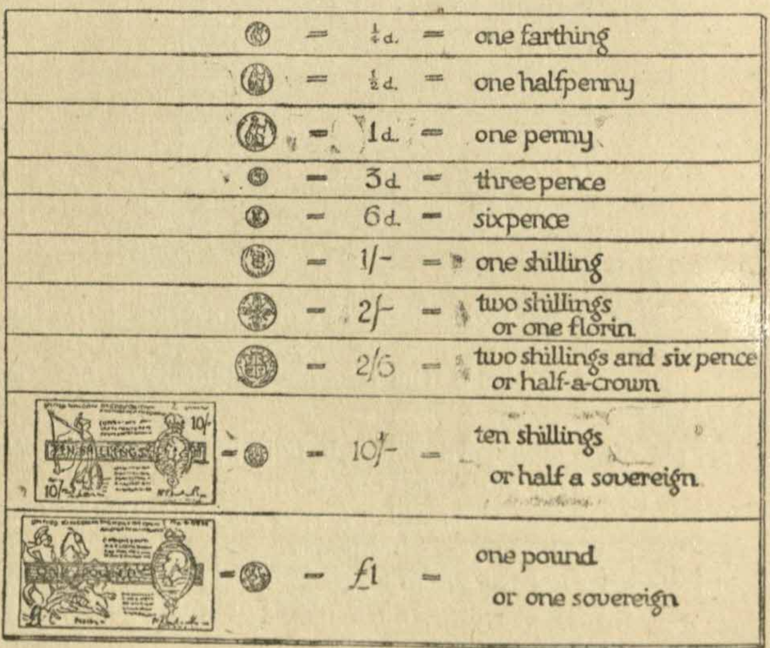


Fig. 148

everyday life so serious that children very soon learn to know them. The best plan is to have a wall-card as in Fig. 148. This not only impresses the 'look' of the coins themselves, but also the spelling of their names and the corresponding signs.

(*a*) *Sorting Coins.* By allowing very small children to sort a box of coins they are led to look carefully at each coin. A

ticket with the name of the coin might be put on each pile, or the coins be placed in labelled compartments.

(*b*) *Matching Coins and Figures.* On cards like that in Fig. 149 the child is required to put the requisite number of coins beside each figure.

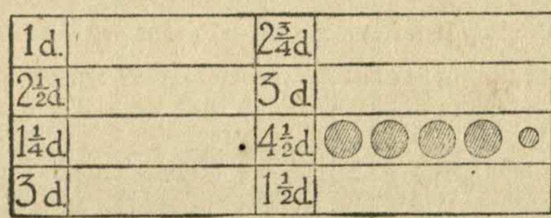


Fig. 149

(*c*) A similar exercise to the above, but priced objects are used instead of simple figures (Fig. 150). These cards should, of course, be graded.

(2) **Relative Value of Coins.** The wall-charts shown in Figs. 148, 152, and 153 are useful for reference.

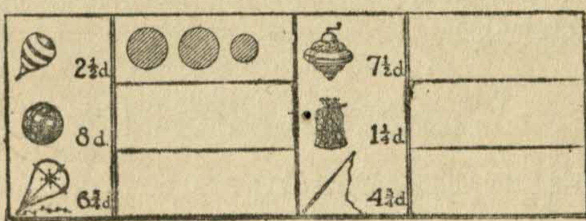


Fig. 150

(*a*) Fig. 151 shows a useful type of card for individual work. The child is required to count out the change in coins, and then to insert digit-tablets for the answer. The cards, which should be graded carefully, might be worked through twice, the second time without coins being used.

(*b*) *Giving Change.* The answers to the following series of exercises are to be given in *coins*:

" Give me change for a shilling with four coins only.

" With an equal number of pennies and halfpennies [*i.e.*, eight of each].

"So that I get no more than 3*d.* in coppers.

"So that I get only three silver coins of two different values.

"So that I can easily pay a debt of $9\frac{1}{2}d.$"

Innumerable other such examples may be invented by the teacher.

(*c*) *Building up* (*but not memorizing*) *Money Tables.* (i) By

Change to Halfpennies	
2 d.–	halfpennies
$3\frac{1}{2}$d.–	"
4 d.–	"
6 d.–	"
$1\frac{1}{2}$d.–	"

Fig. 151

"Welbent" Series. Stage 3, Step 9.

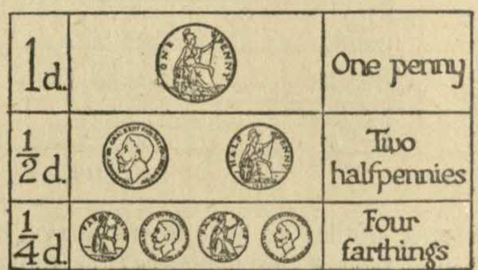


Fig. 152. The Penny Chart

The "Welbent" Money Chart.

using the penny chart (Fig. 152) the child would get the following equations:

$$\begin{aligned} 1 \text{ penny} &= 2 \text{ halfpennies} \\ &= 4 \text{ farthings.} \\ 1 \text{ halfpenny} &= 2 \text{ farthings.} \end{aligned}$$

(ii) Using the shilling chart (Fig. 153):

$$\begin{aligned} 1 \text{ shilling} &= 2 \text{ sixpences} \\ &= 4 \text{ threepences} \\ &= 12 \text{ pennies} \\ &= 24 \text{ halfpennies} \\ &= 48 \text{ farthings.} \end{aligned}$$

(*d*) *Preliminary Exercises with the Money Calculation Boards* (Figs. 144 and 145). (i) It is advisable to work through the cards as follows with the children.

Put farthings in one compartment, counting as you do so "1 farthing, 2 farthings. 2 farthings make 1 halfpenny." Then change for 1 halfpenny, which is put on so as partly to cover the two farthing-marks.

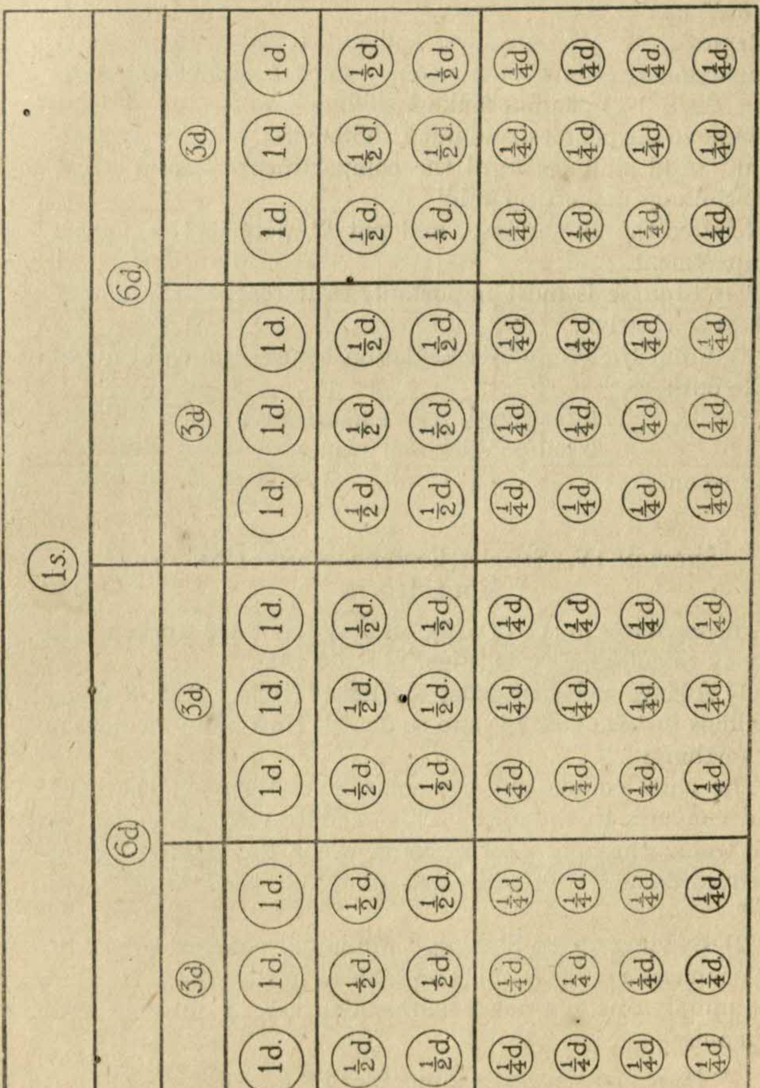


Fig. 153. The Shilling Chart

Add 2 more farthings, change for 1 halfpenny; then, "2 halfpennies make 1 penny."

Remove the 2 halfpennies, and put 1 penny in the next compartment. Count out pennies until a compartment is full; then, "12 pennies make 1 shilling."

Remove the pennies, and put 1 shilling in the next compartment. Add shillings until the compartment is full; then, "20 shillings make 1 pound."

Remove the 20 shillings, and put £1 note in the pounds compartment.

This exercise is most important, as it teaches the money values so clearly.

(ii) From these money calculation boards the child might write out:

2 farthings = 1 halfpenny.
2 halfpennies = 1 penny.

And so on.

SECTION IV. SIMPLE EXERCISES INVOLVING THE FOUR RULES

The exercises in (1), (2), (3), and (4) below are worked with money calculation cards (Figs. 144 and 145).

(1) **Adding and converting to Higher Units.** "How many shillings in 16*d*., 25*d*.?" and so on. "How many pennies in 12 farthings?"

The child counts out the coins, puts them on the card, and converts to shillings. This exercise may be varied by the teacher naming a set of coins, which the children put on their cards and then add—*e.g.*, $\frac{1}{4}d. + \frac{1}{2}d. + \frac{1}{2}d. + 6d. + 5d. = ?$

(2) **Building up the Shilling.** Fill in one compartment with 12 pennies, or arrange coins on table as in Fig. 154. Remove the middle row and put 2 halfpennies instead of each of the 4 pennies.

"How many pennies altogether?"
"How many sets of 3 pence?"
"How many sets of 4 pence?"
"How many $1\frac{1}{2}d$.'s?"

"Count up to 1*s*. (in $1\frac{1}{2}d$.'s); thus, $1\frac{1}{2}d$., 3*d*., $4\frac{1}{2}$d., 6*d*.," and so on.

"How many $1\frac{1}{2}d$.'s in 1*s*. 3*d*., 1*s*. 6*d*., 2*s*. 6*d*.?" and so on.

"If I take $1\frac{1}{2}d$., 3*d*., $4\frac{1}{2}d$., away how much remains?"

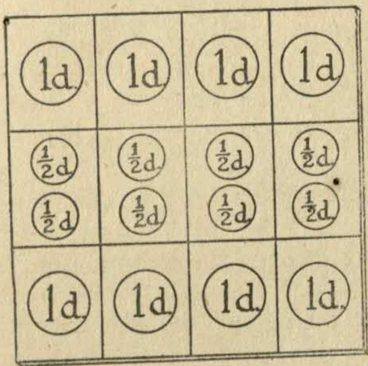


Fig. 154. PENNIES AND HALFPENNIES

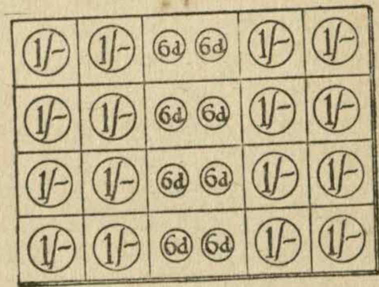


Fig. 155. SHILLINGS AND SIXPENCES

The child might fill in an exercise-card or write equations in a series as

1*s*. = 12*d*.
= 6*d*. × 2 (read as 6*d*. taken twice).
= 4*d*. × 3 (see Chapter VI).
= 3*d*. × 4
= $1\frac{1}{2}d$. × 8.
1*s*. − $1\frac{1}{2}d$. = $10\frac{1}{2}d$.
1*s*. − 3*d*. = 9*d*.

And so on.

(3) **Building up the Pound.** Fill in the shilling compartment, putting 8 sixpences down the middle row (see Fig. 155).

Questions similar to those described under (2) above might be set—*e.g.*,

"Count up to £1 in 2*s*. 6*d*.'s—2*s*. 6*d*., 5*s*., 7*s*. 6*d*."

"Count up to £1 in 1*s*. 3*d*.'s—1*s*. 3*d*., 2*s*. 6*d*., 3*s*. 9*d*."

And so on.

These last two exercises can, of course, be done without cards.

Figs. 156 and 157 show useful exercise-cards.

(4) **More difficult exercises,** such as the following, may be worked from the charts shown in Figs. 144 and 145 :

$\frac{3}{4}d. + \frac{3}{4}d. = 1\frac{1}{2}d.$ $18d. = 1s.\ 6d.$ $18d. \div 2 = 9d.$

$\frac{1}{2}d. \times 5 = 2\frac{1}{2}d.$ $6d. \times 4 = 2s.$ $2s. \div 3 = 8d.$

Make £1
2s.6d.×
1s.3d.×
5s.0d.×
10s.0d.×
1s.0d.×
6d.×

Fig. 156

Make £1
7s.6d.+
2s.6d.+
12s.6d.+
17s.6d.+
8s.6d.+
11s.6d.+

Fig. 157

In the example $18d. \div 2$ the 18 pennies should be either dealt out or shared between the upper and lower compartments or put so as to fill one compartment completely and half the second, groups of two then being counted. (See the chapter on division.) The child should be encouraged to make up his own sums.

(5) **Building up Money Tables from the Number-chart shown in Fig. 147.** The following table might be constructed :

$12d. = 1s.$
$20d. = 1s.\ 8d.$
$24d. = 2s.$
$30d. = 2s.\ 6d.$

And so on.

Shilling pieces might be put on the squares with numbers 12, 24, 36, etc., and the intermediate money values might be written in red ink in the other squares.

(6) **Building up Sums of Money with Priced Cards.** "Progress" Games Nos. 5 to 9—*i.e.,* the 1*s.*, 2*s.* 6*d.*, 5*s.*, 10*s.*,

and £1 games—give a good variety of exercises in simple money calculations.

Game 5, for instance, consists of forty-eight cards, all of values below 1*s*. The following are two of the exercises given:

(*a*) Make up 1*s*. in six different ways, using only two cards each time. (See Fig. 158.)

(*b*) Make up 1*s*. in as many different ways as you can, using three cards each time.

And so on.

There is also an introductory game (No. 4), which does not introduce halfpennies or farthings. As there are four cards of each value 1*d*. to 1*s*., round games such as "Snap" may be played.

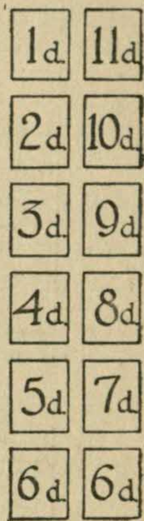


Fig. 158
"Progress" Game 5, "The Shilling"

Section V. Compound Addition

(1) Method of using the Cards for Sums—*e.g.*:

£	*s*.	*d*.
1	14	$6\frac{1}{2}$
2	16	$8\frac{3}{4}$

Put coins on to represent the top row, £1 14*s*. $6\frac{1}{2}$*d*., and below these in separate compartments put coins to represent £2 16*s*. $8\frac{3}{4}$*d*.

(*a*) Push 2 farthings from lower into upper row to complete the 4 that make a penny. Remove the 4 and put a penny in the pence compartment of the upper row. Record $\frac{1}{4}$*d*. with a digit-tablet.

(*b*) Push 5 pennies from lower into upper compartment. Remove the 12 now there and replace by a shilling. Record 3*d*. with a digit-tablet.

(*c*) Push 5 one-shilling pieces up to complete the 20 that make a pound. Remove and replace by a £1 note. Record 11*s*. with digit-tablets.

(*d*) Add the pounds, and record £4 with a digit-tablet. This process is shown in Figs. 159 and 160.

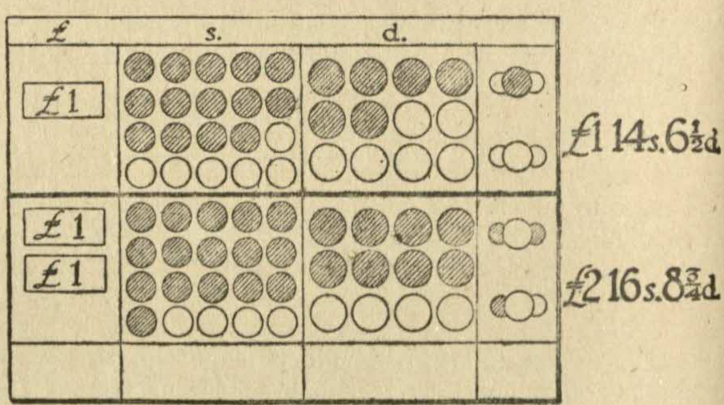


Fig. 159. Card with Coins placed ready for working

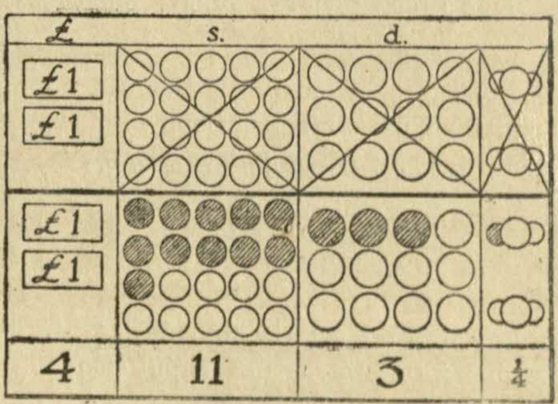


Fig. 160. Card when Sum has been worked

The crosses indicate that those compartments were filled and coins removed and replaced by one coin next in value.

(2) Notes on Compound Addition. Two methods may be used when adding a column of figures :

(*a*) The pence may be made up to shillings as the addition proceeds ; thus there will be no reducing to be done after the pence are added. For instance, the sum 6*d.* + 9*d.* + 10*d.* + 4*d.* would be worked thus : 6*d.* + 9*d.* = 1*s.* 3*d.* ; 1*s.* 3*d.* + 10*d.* = 2*s.* 1*d.* ; 2*s.* 1*d.* + 4*d.* = 2*s.* 5*d.*

Of course, the child would say to himself only "6*d.*, 1*s.* 3*d.*, 2*s.* 1*d.*, 2*s.* 5*d.*"

(*b*) The total number of pence may be found, and then converted to shillings. By this method the above sum would be worked as follows: 6*d.* + 9*d.* + 10*d.* + 4*d.* = 29*d.* = 2*s.* 5*d.* This implies a knowledge of the multiplication tables, and hence for a small child method (*a*) is the easier.

Some teachers advocate the method of allowing the child at first to put down 29 in the pence column and then convert to shillings.

When adding up the shillings column—for instance, in the following:

£	*s.*	*d.*
3	4	8
2	8	6
9	5	10
	16	4

it is best to let the child put down the units figure in the shillings—*i.e.*, 5—at once, as division by 20 will not affect it. The tens carried should be regarded as halves of pounds. In this case £1 would be carried, and the total shillings would be 15.

(See also the chapter on addition.)

Section VI. Compound Subtraction

Use of Money Calculation Cards for Subtraction.

Example:

£	*s.*	*d.*	
4	6	$5\frac{1}{2}$	(minuend)
2	8	$10\frac{1}{4}$	(subtrahend)

(*a*) *Decomposition Method* (see Fig. 161). Put out coins in the upper row of compartments to represent minuend, and put out *digit-tablets* in the lower row to represent subtrahend.

In the farthing column, $\frac{1}{4}$*d.* from $\frac{1}{2}$*d.* leaves $\frac{1}{4}$*d.* Change the $\frac{1}{2}$*d.* coin and put $\frac{1}{4}$*d.* Record the answer with a digit-tablet.

At once it is seen that 10*d.* cannot be taken from 5*d.*, so

change to pence one of the 6 one-shilling pieces from the next compartment. Now 10*d.* from 17*d.* leaves 7*d.* Remove the 10*d.* and record the answer (7*d.*). It will be at once seen that

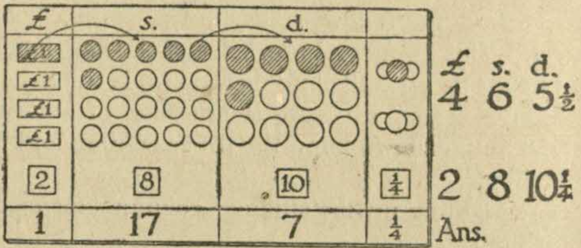


Fig. 161. Subtraction by Decomposition

8*s.* cannot be taken from 5*s.* Change £1 of the £4 into 20 shillings. Now 8*s.* from 25*s.* leaves 17*s.* Record 17*s.* £2 from £3 leaves £1. Record £1.

(*b*) *Equal Additions* (see Fig. 162). This method is based

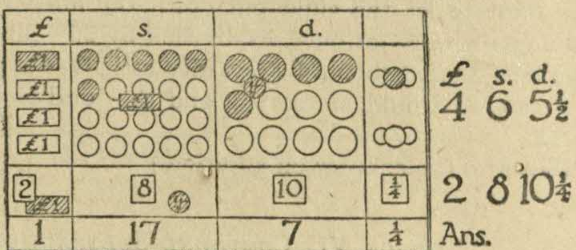


Fig. 162. Subtraction by Equal Additions

on the fact that if equals be added to unequals their difference is unchanged.

Put out coins to represent the minuend and digit-tablets to represent the subtrahend.

In the farthing column, ¼*d.* from ½*d.* leaves ¼*d.* Change the coins and record the answer.

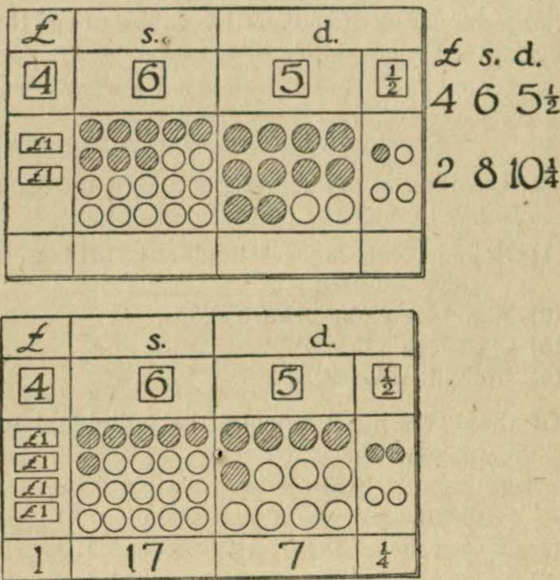
As 10*d.* cannot be taken from 5*d.*, put 12 pennies (shown as 1*s.* in the diagram for the sake of clearness) beside the 5 pennies and put one shilling piece next the digit-tablet 8. Now you have given a shilling to both the minuend and the subtrahend—10*d.* from 17*d.* leaves 7*d.* Record 7*d.*

As 9*s.* cannot be taken from 6*s.*, give £1 in shillings to the minuend and a £1 note to the subtrahend. Then 9*s.* from 26*s.* leaves 17*s.* Record this.

£3 from £4 leaves £1. Record the answer.

See p. 101 (*b*) for alternative wording.

(*c*) *Complementary Addition* (see Figs. 163 and 164). Put out digit-tablets to represent the minuend and coins to represent the subtrahend.



Figs. 163, 164. Subtraction by Complementary Addition—Sum-card before and after Sum has been worked

The aim will be to add coins to the subtrahend until it equals the minuend. In other words, the problem is, " What must I add to £2 8*s.* 10¼*d.* to make £4 6*s.* 5½*d.* ? "

To make ¼*d.* into ½*d.* add ¼*d.* Record ¼*d.* in the answer.

At once it is seen that 10*d.* cannot be made into 5*d.* by addition, hence " What must be added to 10*d.* to make 1*s.* 5*d.* or 17*d.* ? " Add 7*d.* in coins and record it in digit-tablets. Change the 17*d.* into a shilling piece and 5 pennies. There is now 9*s.* instead of 8*s.* in the subtrahend. To make

9*s*. into 26*s*. add 17*s*. Change the 26*s*. into £1 6*s*. Record 17*s*. added.

£3 + £1 = £4. Record £1.

This process has been written out in full to make it quite clear, but, as has been said before, the fewer words the children use while working the better.

The first exercises in complementary addition involving the 'difficulty' might be of the form 3*s*. − 8½*d*., and, later, £3 0*s*. 0*d*. − 7*s*. 4*d*.

For a discussion of the relative values of the three methods see Chapter V.

Section VII. Shopping Activities

The possibilities of shopping activities in schools may best be considered under the following headings :

(1) Make-believe shops with full-sized articles. (Cardboard coins are used.)
(2) Miniature make-believe shops.
(3) Class lessons.
(4) Individual work.

Of these, for most schools, the individual work method will be found most satisfactory. (Actual shops in which transactions with real money take place are not dealt with here, as they are for the upper school only.)

(1) **Make-believe Shops with Full-sized Articles.** (*a*) *One Shop for the Whole School.* An excellent account of a general stores kept in a large boys' school is given in *The Teaching of Arithmetic*, by Potter. No doubt children of nine or ten could conduct such a shop, but it is essential that an enthusiastic, methodical teacher be at the head of things, and that she have great faith in the success of the method and unbounded patience in helping to carry it through in a businesslike way. Such a scheme without careful organization and the whole-hearted co-operation of head-teacher and staff would lead to great waste of time, disorder, and desultory habits of work. Hence it is not every school that could undertake it.

(*b*) *A Class-room Shop.* This is a much easier thing to manage, for the class-teacher alone is responsible. Her great difficulty will be to keep all the children occupied. This might be achieved by having a shop for every eight children, or by giving groups of children turns in going to the class shop, while the others do silent work. It is essential that the scales and weights used should be real ones.

One advantage of such shops is that they give the children some definite aim in their handwork lessons. Many of the things sold could be made by the children. Also the cutting of labels and the pricing should be their own work. Bill-heads should be hectographed by the teacher. The shops, which need not contain many articles, should represent the different trades—*e.g.*, baker, draper, fruiterer, ironmonger, etc. Due regard should be given to current prices. It is a good plan so to organize the shopping that each group of children in turn gets opportunities for weighing out in ounces and pounds, measuring length, etc.

Shopping boxes—*e.g.*, drapery or toys—containing six or seven articles and price-tickets are useful. One could be given between two children. Later sum-cards to go with each box could be supplied. (See (4) below.)

(2) **Miniature Make-believe Shops.** These are very much less useful than the above. Little is learnt by a child of seven or eight in weighing out sand or sawdust into toy scales. When the object of the exercise is merely the calculation of the prices apart from manipulation (such as measuring length or weighing) these little shops may be useful, just as pictures are. They are certainly fascinating as playthings to be used with dolls (for the smaller the child the more delighted it seems to be with anything smaller than itself), but in serious arithmetic we may doubt their value.

(3) **Class Lessons.** These seem useful mainly in introducing a group of children to individual or group exercises. The following are a few suggestions for them :

(*a*) The name of the kind of shop is put up. Objects are arranged on the teacher's table and priced by the children. They then work in pairs ; one child acts as shopman and the other as customer. Each customer takes coins to the value,

let us say, of one shilling. All ‘buy’ the same articles as the child who is at the teacher's table They pay their money, and the shopman calculates, and gives the correct change. The customer checks this change. In the meantime the two children who initiated the activity at the table have finished their transaction under the supervision of the teacher. The results of all can then be checked.

When once the children have grasped the idea they should be allowed more freedom. Those who are able should write bills and receipt them.

(*b*) Pictures pinned to a board or sketches on the blackboard might be used instead of objects. Each child is told to take, for instance, a shilling from his money-box. He has to spend the *whole* shilling, and make out a little bill for himself—*e.g.*:

2 tops for 3*d*.
1 ball for 6*d*.
20 marbles for 3*d*.

Those who work more quickly might see in how many different ways they could spend their shilling.

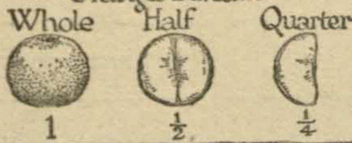
(4) **Individual Work.** (*a*) *Shopping Exercises.* A good series of shopping-cards, through which each child works at his own rate, is the best means of teaching money transactions. Figs. 165–170 show cards taken from such a series; most of these may be used by two children working together. For instance, the card in Fig. 168 is worked as follows.

The shopkeeper reads the list of his goods and then the prices—*e.g.*, “Cabbages at 2*d*. each,” “Tomatoes at 8*d*. a pound,” and so on. (Some words may seem long, but with the aid of the pictures little difficulty should be experienced.)

The buyer then reads the first sum. “8 cabbages at 2*d*. each.” He gives 2*s*., and while the shopkeeper finds the correct change he fills in the answer with digit-tablets. The change is then placed on the card by the shopkeeper, and both check the result by adding the change to the amount spent.

(*b*) *Shopping Boxes and Cards.* The boxes have been described under (1). They are useful, but there is a danger of the articles proving more attractive than the arithmetic!

Oranges 2d. each



6 oranges How many boys can have		How much to pay for	s. d.
$\frac{1}{2}$ an orange	☐ ☐	1 dozen oranges	☐ ☐
$1\frac{1}{2}$ oranges	☐	6 oranges	
2 whole oranges	☐	$\frac{1}{4}$ dozen oranges	
$\frac{1}{4}$ of an orange	☐ ☐	3 dozen oranges	
3 whole oranges	☐	6 half-oranges	

Fig. 165

"Welbent" Series, Stage 5, Step 10.

1 dozen sticks of rock

2/- a dozen

	s. d.
$\frac{1}{2}$ dozen sticks cost	☐ ☐
3 sticks cost	d.
4 sticks cost	d.
$\frac{1}{4}$ dozen sticks cost	d.
1 stick costs	d.
$\frac{1}{2}$ stick costs	d.

Fig. 166

"Welbent" Series, Stage 5 Step 10.

Flour 2lb. bag costs 8d.

16oz. 1lb.
8oz. $\frac{1}{2}$lb.
4oz. $\frac{1}{4}$lb.

o. means ounce or ounces
lb. means pound or pounds

	s. d.
1lb. of flour costs	☐
4lb. of flour cost	
6lb. of flour cost	
8oz. of flour cost	
$\frac{3}{4}$lb. of flour costs	
$3\frac{1}{2}$lb. of flour cost	

Fig. 167

"Welbent" Series, Stage 5, Step 10

What is the cost of:	s. d.	Change from 2/-
8 cabbages at 2d. each		
3lb. tomatoes at 8d. lb.		
6lb. potatoes at 3d. lb.		
4 bunches leeks at 5d. a bunch		
4 marrows at $4\frac{1}{2}$d. each		
7lb. carrots at $1\frac{1}{2}$d. a lb.		

Fig. 168
"Welbent" Series, Stage 5, Step 11.

Oranges for sale
1/6 a dozen

$\frac{1}{2}$ dozen oranges cost	
$\frac{1}{4}$ dozen oranges cost	
Bess wants a dozen oranges in the basket. Bess must add	oranges
Ted buys 2 dozen oranges at 1/6 a dozen. Ted pays	
He sells $\frac{1}{4}$ of his oranges to Nan. Nan must pay him	
Mum has 5 boys. She gives each 4 oranges. Mum gives	oranges

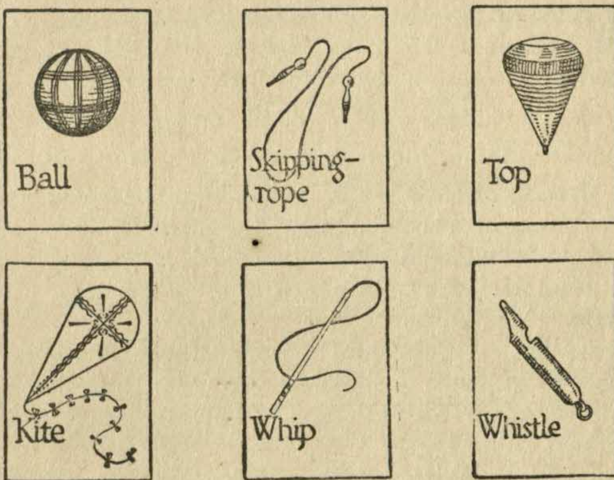
Fig. 169
"Welbent" Series, Stage 5, Step 11.

To spend

Reels of cotton at $1\frac{1}{2}$ d. each		Hat-pins at 2d. each	
Balls of wool at 3 d. each		Scarves at 8d. each	
Cards of pins at 2d. a card		Belts at 6d. each	
Packets of needles at 1d. each		Caps at 2/- each	
Pairs of boot-laces at $\frac{1}{2}$ d. a pair		Bonnets at 1/- each	
Pairs of socks at 1/- a pair		Handkerchiefs at 4d. each	

Fig. 170
"Welbent" Series, Stage 5, Step 11.

Envelopes with sets of pictures[1] and price-tickets. and exercise-cards are also useful (see Fig. 171), but they are not



6d 10d 3d $7\frac{1}{2}$d 1/- 2d

Put a price-ticket on each little card Then fill in this card with digit-tablets	s.	d.
Cost of 5 whistles		
Cost of 2 kites and 1 ball		
Cost of 4 skipping-ropes		
Cost of 3 tops and 3 whips		
Cost of 6 balls and 2 kites		
Cost of 2 whistles and 3 whips		

Fig. 171

so practical as the shopping-cards, for the loose pieces take up more room on the desk and are liable to get lost.

[1] The pictures from the "Pathway Picture Scheme" (Robert Gibson) are quite excellent for this purpose.

Section VIII. Compound Multiplication

(1) **Introductory Exercise to show that Multiplication is Continued Addition.** Examples such as the following may be worked with coins on the money calculation cards:

(*a*) $4\frac{1}{4}d. + 4\frac{1}{4}d. + 4\frac{1}{4}d. + 4\frac{1}{4}d.$, or $4\frac{1}{4}d. \times 4$.

(*b*) $5s.\ 2d. + 5s.\ 2d. + 5s.\ 2d. + 5s.\ 2d.$, or $5s.\ 2d. \times 4$.

(2) **Grading of Early Work.** Coins should be used for the first steps. Money abaci can be had at 4*s*. 6*d*., but they seem hardly necessary if the money calculation cards have been used.

(*a*) *With Multipliers not exceeding* 6. *First Step.* With no reduction—*e.g.*, $2s.\ 3d. \times 3$.

Second Step. With reduction of halfpennies and farthings to pence or of pence to shillings—*e.g.*, $4\frac{1}{2}d. \times 2$; $8d. \times 3$.

Third Step. With reduction of both farthings and pence —*e.g.*, $9\frac{1}{4}d. \times 6$.

Fourth Step. With reduction of both farthings and pence, and with shillings introduced—*e.g.*, $2s.\ 6\frac{1}{2}d. \times 3$.

Fifth Step. With reduction of shillings to pounds—*e.g.*, $8s.\ 8\frac{1}{2}d. \times 4$.

Sixth Step. With reduction of shillings to pounds, and with pounds introduced—*e.g.*, £2 $9s.\ 7\frac{1}{4}d. \times 5$.

(*b*) *With Multipliers under* 12. The same grading might be used as in (*a*).

The child should, of course, multiply by 11 and 12 in one step.

For all these examples it is essential that the multiplication tables should be well known.

(3) **Factor Multiplication.** (*a*) *Examples in which the Multipliers are easily factorized.*

Example: £2 $9s.\ 8d. \times 18, 24, 25$.

These should present little difficulty.

(*b*) *Examples in which the Multipliers cannot be factorized directly, but are closely allied to Factorizable Numbers.*

Example: £6 $7s.\ 3\frac{1}{2}d. \times 41$ or 34.

41 may be taken as $8 \times 5 + 1$ or as $4 \times 10 + 1$.

34 may be taken as $7 \times 5 - 1$ or as $8 \times 4 + 2$.

The example £6 7*s.* 3½*d.* × 43 would be set down as follows:

£	*s.*	*d.*	
6	7	3½	
		7	
44	11	0½	(7 times)
		6	
267	6	3	(42 times)
6	7	3½	
273	13	6½	(43 times)

There is, however, a danger of this method being pushed too far by a child. In his anxiety to avoid multiplying by a difficult number he is apt to waste time in finding suitable easy combinations, and not infrequently inaccuracies arise in the working which are not easy to trace. Hence teachers are strongly advised to train their children to do long compound multiplication, which is of universal application.

(4) **Beginnings of Formal Compound Multiplication.** It is essential that the children should be clear about the commutative law. They should not be troubled by long explanations about abstract and concrete quantities. It will be sufficient if they realize that, for instance, 8*d.* × 23—*i.e.*, 8*d.* taken 23 times—is the same as 23*d.* × 8, or 23*d.* taken 8 times. They should be led to see the advantage of taking the smaller number as multiplier. It does not really signify whether the multiplier is placed above or below the multiplicand in the actual setting down of the sum. The two methods are shown in (*a*) and (*b*) below.

Examples:

(*a*)	8*d.* × 23.	19*s.* × 23.	£21 × 23.
	8*d.*	19*s.*	£21
	23	23	23
	12)184*d.*	230	460
	15*s.* 4*d.*	207	23
		20)437*s.*	£483
		£21 17*s.*	

```
(b)        23d.            23s.          £23
            8              19            21
      12)184d.             230           460
         15s. 4d.          207            23
                        20)437s.        £483
                           £21 17s.
```

After working a number of examples of this description the child should be ready for examples involving more than one denomination in the multiplicand.

(5) More Difficult Long Compound Multiplication.

Example:

(*a*) £29 14*s*. $6\frac{3}{4}$*d*. × 37.

```
         £         s.        d.
         29        14        6¾
                             37
       £1099       18        9¾

         26        20        27        4)111 farthings
        740       370       222           27d. 3 f.
        333       148    12)249
       £1099   20)538        20s. 9d.
                 £26 18s.
```

This arrangement is clear and compact, and much to be preferred to that shown in (*b*). Its chief disadvantage is that the multiplier 37 is rather far from the multiplicand £29. This sometimes causes inaccuracies in a child's work. At first, therefore, he might be allowed to write the 37 again in pencil under the £29.

```
         £         s.        d.
(b)      37        37        37
         29        14        6¾
       £1099       18        9¾

         26        20        27        4)111 farthings
        740       370       222           27d. 3 f.
        333       148    12)249
       £1099   20)538        20s. 9d.
                 £26 18s.
```

Here the 37, originally the multiplier and the abstract quantity, has been set down as the multiplicand and the concrete quantity. Hence there are three distinct multipliers —*e.g.*, $6\frac{3}{4}$, 14, and 29. This arrangement is not recommended, as it is difficult for children to understand.

Its only advantages are that the multiplier and multiplicand are close together, and that the larger number, 37, is at the top. This, however, would not be the case if the number of pounds were greater than 37. Moreover, the relative positions of multiplier and multiplicand (provided they are near together) do not increase the difficulty of the process.

(6) **Practice.** Some teachers prefer to make the children do all their compound multiplication sums by practice. In cases where the fractional parts are simple and easy to find this is an excellent method. In fact, for mental work it is the most natural to use.

Section IX. Compound Division

(1) **Introductory Exercises.** With the money calculation cards, examples such as the following should be worked:

"How many threepences can I take from 2*s.* ?"

To work this both pence compartments are filled.

"How many times must I take $1\frac{1}{2}$*d.* to make 9*d.* ?"

"How many times is 2*s.* contained in 18*s.* ?"

"If 6*s.* 9*d.* is divided between three people, how much will each get ?"

(For a discussion on the different aspects of division the reader is referred to Chapter VII.)

(2) **Beginnings in Formal Compound Division.** (*a*) The method used for compound division should be, as far as possible, the same as that used for simple division. No elaborate explanations need be given as to the difference between partition and quotition, nor between concrete and abstract quantities. The child should be taught to adapt his method to the sum in question.

In Fig. 172 a simple example worked in coins is shown. In this particular example partition is used. The various coins are dealt out to each sharer and placed in vertical lines —" One shilling to John, one to Jack, one to James ; two to John, two to Jack," and so on.

Fig. 172

7*s.* 1½*d.* to be divided equally among three boys—John, Jack, and James.
Deal out 1 shilling to each, then another to each. Change the remaining shilling for 12 pennies.
Deal out the pennies. Change the remaining one for 2 halfpennies.
Deal out the halfpennies.

Examples such as the following should be worked in coins :

(i) 6*s.* 8*d.* ÷ 5.
(ii) 16*s.* 10*d.* ÷ 4.
(iii) 17*s.* ÷ 8.
(iv) £1 12*s.* 6*d.* ÷ 10.

The child should not continue too long saying " One to A, one to B," and so on, but should, for instance, in (i) deal out the shillings in a vertical column of five, then convert the remaining shilling to pence and deal them out in fives too.

The grading of the work should be the same as for multiplication.

(*b*) *Dividing Money by Money.* Here we are dealing with successive subtractions. Fig. 173 shows a simple sum worked in coins. The child would probably have reduced the shillings to pence instead of to threepences. This example would, therefore, be a good one to use to show how the process may be shortened by reducing to the largest amount possible. It is not advisable to give the child a number of difficult exer-

cises to be worked with coins. After one or two have been gone through he should do them in his head, and only simple ones such as he will meet with in his shopping exercises should be given. Later on he will work more complicated ones by fractions or decimals, and hence time should not be

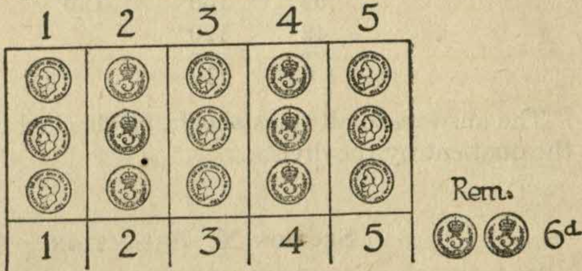


Fig. 173

To how many boys can I give 9*d*. if I have 4*s*. 3*d*. ?
Change 4*s*. 3*d*. into threepenny-pieces. Arrange in vertical groups of three.
5 boys can get 9*d*. each.
There are two threepenny-pieces over.

spent on them at this stage. Examples that might be worked are 22*s*. 6*d*. ÷ 2*s*. 6*d*., 18*s*. ÷ 1*s*. 6*d*.

The child should be encouraged to construct his own problems around these numbers, as, for instance, "How many spinning tops at 1*s*. 6*d*. each can I buy for 18*s*. ? "

(*c*) *N.B. Factor Division.* On account of the difficulty of remainders this method should not be taught to children under eleven ; hence it is beyond the scope of this book.

(3) **Long Compound Division.** The arrangement shown below is advocated. The whole process is not nearly so difficult as it looks. If children are given carefully graded examples, and if they have been taught a similar arrangement for long compound multiplication, this will not appear difficult to them.

(*a*) £43 7*s*. 8*d*. ÷ 6.

£7	4*s*.	$7\frac{1}{3}$*d*.	*Ans.*
6)£43	7*s*.	8*d*.	
42	20	36	
1	27	44	
	24	42	
	3	2	

(*b*) £294 10*s*. ÷ 12.

	£24	10*s*.	10*d*.	*Ans.*
12)	£294	10*s*.	0*d*.	
	24	120	120	
	54	130	120	
	48	120		
	6	10		

The answers to all sums should be checked by multiplying the quotient by the divisor.

Section X. Reduction

Reduction may imply either division or multiplication, for we say "Reduce 1*s*. 6*d*. to farthings" and "Reduce 24 farthings to pence." The word 'change' would be better, as it expresses both operations more simply. Many of the 'reduction' sums met with in our arithmetics are quite absurd. For instance, in real life it is seldom necessary to reduce 12,000,000 in. to miles.

The following points should be borne in mind while teaching reduction:

(1) The examples should be such as would be met with in real life.

(2) They should be carefully graded—those involving one step only coming first, then those with two, and finally those with more than two steps.

(3) The child should be made to realize that the two types of reduction involve reverse processes, and should be encouraged to check by division results obtained by multiplication. Children sometimes distinguish the two operations by saying "Reducing *up* by dividing" and "Reducing *down* by multiplying."

(4) Problems should be given. They are extremely easy to construct.

CHAPTER IX

WEIGHTS AND MEASURES

SECTION I. NECESSITY OF PRACTICAL METHODS

IN no part of the syllabus is practical[1] work more necessary than when the different weights and measures are being dealt with, for the main reason of teaching weights and measures lies in their practical value. Hence the measuring and weighing done by the children should have an aim and require accuracy; the measurements of weights and prices given should be sensible and in accordance with true value in real life. The examples in arithmetic text-books are not sufficiently living to give adequate training. The teacher must devise exercises in connexion with handwork, needlework, gardening, or carpentry to make the weights and measures real quantities to the children.

The most valuable types of problems for practical work are those that not only deal with real life, but which require the child to find his own data. For instance, a box and specimen of priced wallpaper might be provided, and the child asked, "How much would it cost to cover this box with this paper?"

SECTION II. BRITISH UNITS OF LENGTH

History.[2] The smallest measure that the Anglo-Saxons used was the length of a *barleycorn*. Three barleycorns placed end to end or six placed side by side make one inch.

The Roman *uncia*, or *inch*, was the distance between the joint and tip of the longest finger.

[1] For a good discussion of the meaning of the term 'practical' see Potter, *The Teaching of Arithmetic*, pp. 127–130.

[2] The matter for the history of weights and measures has been taken chiefly from Watson's *British Weights and Measures*, S. Cunnington's *The Story of Arithmetic*, Webster's *Dictionary*, and the *Encyclopædia Britannica*.

A *thumb's breadth* was also equivalent to an inch, and three thumb's breadths made a *palm* or *hand's breadth.*

A *cubit,* or *ulna* (Latin *ulna,* 'elbow'), was the length of a man's arm from the elbow to the end of the middle finger.

The cubit was later taken to equal half a yard, or eighteen inches, so that a palm or hand's breadth was a sixth of a cubit.

The *foot,* first taken from the length of a man's foot, was found to equal twelve thumb's breadths or thirty-six barleycorns. The ancient Roman foot is supposed to have been about 11·65 English inches.

A Roman *pace* was a double step. The regulation Army step is now thirty inches ; hence a pace would be sixty inches or five feet.

Now, however, we use pace as synonymous with step.

The *furlong,* or *furrow-length,* is equal to forty perches or 220 yards. The acre's breadth is equal to four perches or twenty-two yards. This last length is now called a *chain* (see p. 186 for notes on Gunter's chain).

The *mile* signified 1000 paces (*mille passus*)—*i.e.*, 2000 steps.

The *yard* is now our standard measure.

" The length of the imperial standard is preserved in the form of a cylindrical bar of platinum supported on metal rollers of a different metal, and with two small plugs of gold sunk in the bar so that their diameters are exactly 36 in. apart. The temperature of the chamber in which the bar is kept should be 60° Fahrenheit ; otherwise there may be some expansion or contraction of the metal.

" The actual standard yard is deposited in the new Record Office in Chancery Lane, and copies of it are kept at the Royal Mint, the Royal Observatory (Greenwich), and the office of the Royal Society." [1]

" In British statutes the yard is termed ulna, aune, aulne, alne, verge, yerde, yard." [2]

Rod, Perch, or Pole. In our present table of lengths we have one perch, rod, or pole equal to five and a half yards. This

[1] Cunnington, *The Story of Arithmetic* (Allen and Unwin).
[2] See Watson, *British Weights and Measures* (J. Murray), p. 58.

was, of course, obtained from the acre's breadth (22 yds.), of which it is a quarter.

(1) **Exercises in Measurements of Length.** The first thing to bring home to the children is the need for a fixed unit, and this applies not only to measures of length, but to all weights and measures. This may be done by allowing the children to measure some length with different objects, and by comparing the answers to lead them to see that the magnitude of the result depends on the length of the thing used to measure with. The conclusion would be, "Unless we all measure this object by the same length we shall get different answers."

(2) **Building up of the Table. Introduction to the Inch, Foot, and Yard.** (*a*) The first pieces of apparatus given to a child should be plain strips of cardboard—or, better still, wooden rods—one inch, one foot, and one yard in length. These should be introduced one by one. The foot could be used for measuring the table, desk, and blackboard, the yard for rolls of ribbon, tape, etc., and the inch for smaller articles. Only after doing this practical work should the child be required to compare the three lengths and build up the table himself.

(*b*) *Estimating Lengths.* It always interests children to find the different lengths on themselves. Thus an inch will often be the length of finger or thumb to the first joint, and a 'span' from the tip of the thumb to the tip of the middle finger six inches or half a foot. They should be allowed first to estimate a length by sight, then to measure it with their hands only, and finally to check these estimates by using a ruler or tape-measure.

(*c*) *Exercises with a Graduated Ruler or with the Foot-box.* These might be similar to those described in the chapter on fractions. In any case, the children should early be taught to deal in quarter and half inches, feet, and yards.

(*d*) *Measurement of Longer Distances.* A ball of thick waxed string may be divided off into yards and feet by the children sewing in and tying pieces of coloured wool. (Beads are good for this, but it is more difficult to get the spacing accurate.) With such a piece of apparatus the garden or playground may be measured. A wheel such as one can buy

in toyshops is another useful piece of apparatus (see Fig. 174). The circumference of the wheel is measured and graduated. A nail and piece of tinfoil are attached near the axle in such a way that a distinct click is heard every time the nail passes the tinfoil—that is, every time the wheel performs one complete rotation. Children love to go walks with such a wheel and to count the clicks. Then on returning home there is a multiplication sum to be done!

Counting steps when walking is another rough method for calculating distance. Legs may be tied together just above the ankle in such a way that when the string is taut the step is some convenient length. If the feet are not thus tied the *average* step of the child must be found.

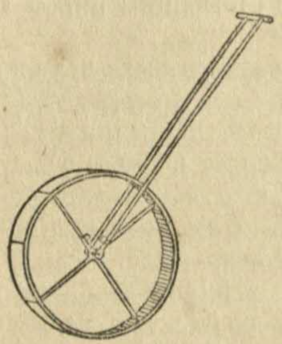


Fig. 174

To give children some idea of a furlong and a mile they should measure one out—even though the playground is not very large it is stimulating to know that by walking several times round it one has walked a mile!

For this, too, the teacher should have a map of the roads in the neighbourhood of the school and should suggest or let the elder children find mile, two-mile, or three-mile walks. The *time* taken to walk a mile should be calculated, and then long distances may be expressed by time—*e.g.*, "It is an hour's walk for me" might mean for a child that it was two to two and a half miles.

If a surveyor's chain can be borrowed it is worth showing the children. A chain is twenty-two yards—an acre's breadth—a familiar measurement to boys, as it gives the length of a cricket-pitch.

From the chain a furlong or furrow's length can be built up. Ten chains equal one furlong. Finally, eight furlongs go to the mile.

If the rod, pole, or perch is to be taught at all, a rod or pole five and a half yards long ought to be provided, and its connexion with a chain should be impressed on the children.

A line twenty-two yards long marked on a wall of the playground or on the ground and divided into poles and feet is most useful, for then the children not only visualize the distance, but can walk and measure strings along it.

Another favourite exercise with children may be mentioned —estimating the heights of houses by counting the bricks. This helps them later to realize heights of mountains.

(3) **Suggestions for Problems and for carrying out Practical Work.** The following is a good plan. Special problems are prepared by the teacher based on measurements taken in and around the school. Each problem is on a separate card, and should involve a considerable amount of work. The teacher would of course keep the answers. As these problems necessitate the children leaving their places, and it may not be possible to allow the whole class to do this at the same time, there should be a further supply of practical problems which could be worked at a desk. The apparatus for these might consist of a box or envelope containing the necessary articles, with the problem clearly printed on the outside. Miss Thomson[1] suggests the following :

" New lace has to be provided for the white tray-cloth ; it is put on plain, with fullness only at the corners. How much lace must you buy, if you allow 3 inches extra at each corner for gatherings and 1 inch for joining ? Find the cost at $5\frac{1}{2}d.$ a yard."

An actual tray-cloth and tape-measure would be put in an envelope or box, with the question clearly written on the outside. If the problems are carefully chosen and the children are put into small groups this practical work does not involve any waste of time, and it serves to make the work more interesting.

Section III. Measures of Weight

(1) **History.** The history of the gradual growth of our present table of weights is very long and complicated. The facts given below have been chosen as those most likely to interest children if they are somewhat simplified by the teacher.

The Pound. In ancient times money was weighed. The

[1] *The Art of Teaching Arithmetic* (Longmans, Green and Co.).

pieces were often also counted, but they were of different shapes and sizes. The ancient Roman pound in weight of silver represented a definite amount of money. Thus the word 'pound' came to represent a money value and a weight. At first these were thus definitely connected, but as time went on the number of shillings coined from a pound of silver increased from twenty to twenty-five or thirty—that is, the value of each coin diminished and the merchant pound weight increased in value—so that the pound in money ceased to have any connexion with the pound in weight.

The word 'pound' means weight. Our abbreviations lb. and £ come from the Latin word *libra* (plural, *libræ*), which means "a balance."

Troy and Avoirdupois. The troy measures are only used now in buying and selling gold, precious stones and drugs. The word 'troy' did not originally mean a particular kind of pound, nor did it refer to the nature of the article weighed, but to the manner of weighing. 'Troy' is probably derived from the old English word 'troi,' signifying a balance. Another form of the word was 'tron,' used in parts of England and in Scotland. 'Tron' was also used to express the market, or place of weighing, and still exists in Scotland in words such as 'Trongate.' . . . The nature of the articles weighed by troy were those of which the value was considerable relatively to the weight.[1]

The word 'avoirdupois,' on the other hand, was not the name of a particular kind of pound, but was a generic word, used with respect to articles of which the weight was considerable relatively to the value, such as iron, wool, lead, etc.[2]

Ounces. The Roman pound weight was divided into 12 *unciæ*. Hence our word "ounce." Later the avoirdupois pound was introduced for heavier goods and subdivided for convenience in "continued halving" into 16 ounces.

Grains. It is interesting that the grain should have been used for weighing as well as for measuring length. At first the wheat grain and then the barley grain was used.

The *hundredweight* was originally equal to one hundred pounds. In our abbreviation cwt. the *c* stands for *centum*,

[1] The present writer would point out that according to the *New English Dictionary* the "received opinion" is that the term is derived from the name of a weight used at the fair of Troyes in France.

[2] Watson's *British Weights and Measures* (J. Murray).

"one hundred," *wt.* for weight. Just as a baker gives 13 buns when one dozen are ordered, so the wholesale merchant gave extra pounds when a hundred were wanted. The hundred-weight varied in value from 108 to 120 lb. at different periods in history and also in different localities. From about the reign of Elizabeth it has been fixed at 112 lb.

The *quarter* is a quarter of a hundredweight—or 28 lb.

The *stone* is an eighth of a hundredweight—or 14 lb.

The *ton* is probably taken from the Old English *tun* or *tume*, a large-sized cask, hence a very heavy one.

The *dram* is a shortened form of drachma, which means 'grasp' or 'handful'—a dram is measured in grains.

(2) **Exercises in Weighing.** (*a*) *To find the Number of Ounces in a Pound.* A 1-oz. and a 1-lb. weight are provided. The child weighs out ounces of sand, putting each ounce into a separate bag. Then he finds how many such bags he requires to balance the 1-lb. weight. Should this exercise be considered too long, the 1-oz. bags of sand might be provided. The values of $\frac{1}{2}$ lb., $\frac{1}{4}$ lb., and $\frac{1}{8}$ lb. should also be found with the 1-oz. bags.

An exercise-card such as that shown in Fig. 167 should be worked through after this practical exercise.

(*b*) *To show that 1-lb. Weights of Different Substances have Different Volumes.* Sawdust, bran, sand, beans, flour, pebbles, etc., may be measured out and done up in 1-lb. bags. Packets of tea, cocoa, etc., bought in shops may also be used.

It might be demonstrated that while a pound of wood will float on water, a pound of shot will sink. The theory underlying all this is too difficult for small children, but these experiences will be found useful in later work in specific gravity, etc.

(*c*) *Making up Sets of Weights.* To enable any number of ounces up to 8 oz. to be weighed out only four weights are needed. The children might be required to find which weights are needed and to construct them with sand. Similar sets in quarter- and half-ounces and for larger quantities might be constructed.

(*d*) *Weighing parcels and letters* for the post and calculating the price.

(*e*) *Dividing up 2- or 3-lb. Packets into Smaller Packets.* For example, exactly 2 lb. of bran are given to a child, and he is told to divide and do it up into ½-lb. packets. No weights are provided.

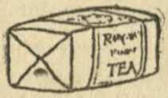


Fig. 175. 1 Lb. Tea (16 Oz.)

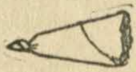


Fig. 176. 1 Oz. of Sweets

Fig. 177. 1 Stone of Flour (14 Lb.)

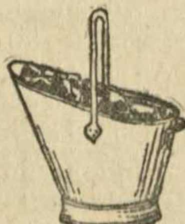


Fig. 178. Quarter-cwt. of Coal (2 Stones, or 28 Lb.)

Fig. 179. 1 Cwt. of Coal (4 Quarters, or 112 Lb.)

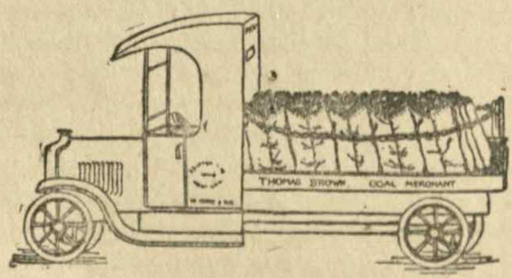


Fig. 180. 1 Ton of Coal (20 Cwt.)

(*f*) *Weighing Larger Quantities.* As it is not possible to give practice with heavier weights, all that can be done is to make the children familiar with some commodity sold in large quantities and to give them some idea of the magnitude of, for instance, a hundredweight and a ton. For a ton, a truck or cart of coal is familiar to most children. Tons of potatoes, clay, etc., are also of common experience. (See Figs. 177–180.)

A hundredweight of clay is often bought for school use, and is useful for giving the children an idea of the magnitude of a hundredweight. Even where it is possible it would hardly be worth while to let them weigh it out, though it might interest them to know that done up in 1-lb. packets there would not be enough clay for a hundred and twelve! This is, of course, accounted for by the fact that it takes a little extra to 'turn' the balance, and this extra quantity becomes appreciable only when there has been a great number of weighings.

The children should weigh themselves frequently, and each one should keep a record of the variations in his own weight from month to month. The magnitude of a ton is brought home to them by adding their own weights and finding how many children are needed to balance a quarter, half, or one whole ton.

The weekly increase in weight of a puppy or kitten gives opportunity for drawing an interesting and simple graph.

(3) **Suggestions for Problems and Practical Work.** Every school should possess a machine for weighing the children, and a strong pair of scales such as are used in grocers' shops. Besides these, other types of weighing machines—for instance, kitchen spring-balance, letter-balance, simple school or home-made lever-balances—should be available. The problems given to be worked should be connected with those on capacity. They should be on cards, in envelopes, or in boxes as suggested for problems on length, but as a general rule it will be quite impossible for a whole class to do weighing at the same time. Even where there is a good science room the balances are generally too delicate and the pans too small for the type of work that has to be done. Hence the children must take turns at using the available balances, and problems on other subjects must be given to the rest of the class.

Frequently children like to take a problem involving practical work home with them. This kind of homework is excellent, for it enlists the help of the parents and is not too strenuous for the child.

In some schools home-made lever-balances are used, and every child has one. From their seesaws children know

something of the principle of the balance, and to make a rough balance and weights is an excellent exercise, but here, again, it is best done as homework.

Problems of the following kinds are recommended:

(*a*) Here are 6 pennies. Weigh all 6 together, then weigh 3 of them.

How much would 3 shillings' worth, 5 shillings' worth, 10 shillings' worth of pennies weigh?

If a handful of pennies weighs 6 oz., how many pennies are there?

(*b*) Here is a bottle. Find out what weight of water it will hold.

(*c*) What would it cost to send these three books by post? Would it be cheaper to send them in two parcels or in one?

SECTION IV. MEASURES OF CAPACITY

(1) **History.** The original idea of measure of capacity was based on weight. A pint measure was a vessel which held a pound weight of wheat. As the specific gravities of the substances measured differed, had this idea of equality in weight been adhered to a great many different-sized pints and gallons would have had to be constructed. In 1824 the standard or imperial gallon was defined as the volume of ten avoirdupois pounds of distilled water at the temperature of 62° Fahrenheit with the barometer at thirty inches.

The original corn bushel contained sixty-four troy pounds of wheat. At the present day the only legal bushel contains eight imperial gallons or eighty pounds of distilled water.

(2) **Exercises in Measuring Capacity.** (*a*) *Building up the Table for Liquid Measure.* The child should be provided with a pint and a quart milk-bottle and a gallon tin. These should be clearly labelled on the outside. By using water the child will find out the following facts:

2 pints = 1 quart.
4 quarts = 1 gallon.

The fact that " quart " stands for " quarter " should be noticed, as in a table with alternating twos and fours this is a help.

Sand or grain ought not to be used here, but later on when the table for dry measure is learnt.

(*b*) *Finding and testing the Capacities of Various Vessels.* A variety of bottles and jars should be provided—*e.g.*, a gill jar sold with cream, an ordinary tumbler, ¼-pint and ½-pint milk measures, a milk pail, stone ink-bottles, wine-bottles, petrol-tin, etc. The exercises may be put into problem form —*e.g.*, " Here is a gallon of milk. To how many children can you give a tumblerful ? "

(*c*) *To build up the Table for Dry Measure.* Pint, quart, gallon, peck, and bushel measures should be provided. The last two are very easy to construct from boxes, tins, or baskets. To build up the lower part of the table the child should use fine sand or grain. For the pecks and bushels Indian corn, acorns, chestnuts, the short grass clipped by a mowing machine, or paper torn into small pieces will do very well.

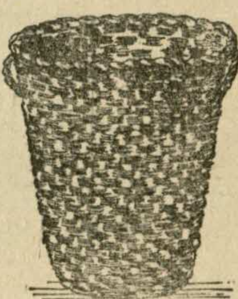


FIG. 181. 1 BUSHEL

4 gills = 1 pint.
2 pints = 1 quart.
4 quarts = 1 gallon.
2 gallons = 1 peck.
4 pecks = 1 bushel.

(See Figs. 181–186.)

(*d*) *Estimating Capacity and Weight.* The children should be encouraged to estimate the capacity of a vessel by eye, and then by rough measurements. Finally they should check their results by accurate methods.

Weight is more difficult to estimate than either length or capacity. Comparison is nearly always necessary. Thus a child might be expected to feel which is the heavier of two objects. If a 1-lb. weight is used, rough approximations may be made, but too much time should not be spent on this.

(*e*) *Finding Relations between Measures of Capacity and Weight.* The fact that

A pint of clear water
Weighs a pound and a quarter

should be found out by the children. This is easily done by counterpoising a vessel capable of holding more than a pint

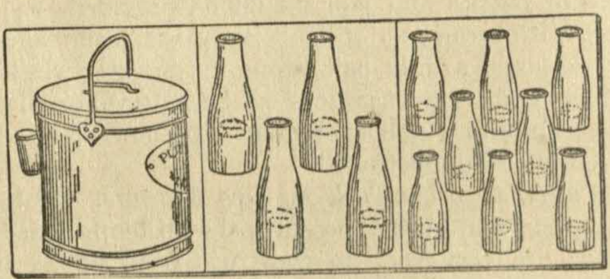


Fig. 182. 1 GALLON = 4 QUARTS = 8 PINTS

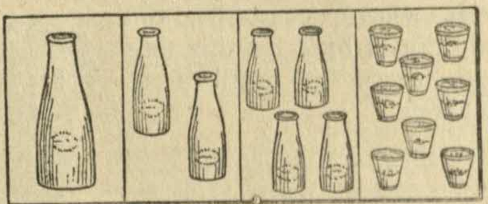


Fig. 183. 1 QUART = 2 PINTS = 4 HALF-PINTS = 8 GILLS

Fig. 184. 1 QUART

Fig. 185. 1 GALLON

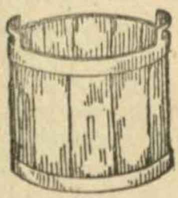


Fig. 186. 1 PECK

of water and then pouring in water from a pint measure and adding weights.

It is also instructive to let children weigh out pints of different substances, such as sand, sawdust, bran. This prepares them for later work in specific gravity, etc.

(3) **Exercises in finding Cubical Contents.** The work should be done on the same lines as for area (Section V). The child should build large cubes and rectangular blocks with cubic inch blocks. These are provided in quantities in most infant schools.

Montessori's bead squares and cubes are dealt with in the chapter on number-study. They are useful for squaring and cubing *numbers*, but should not be used for *volume.*

SECTION V. MEASURES OF AREA

(1) **History.** The square inch, foot, yard, and mile arose out of the measures of length. The British acre, however, has an interesting history, arising as it does from the activities of the people.

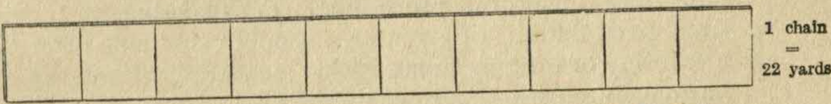


Fig. 187. THE ACRE (1 FURLONG = 10 CHAINS = 220 YARDS)

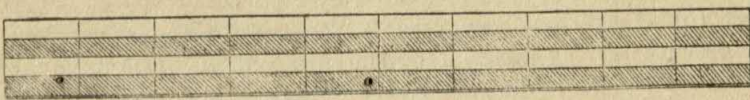
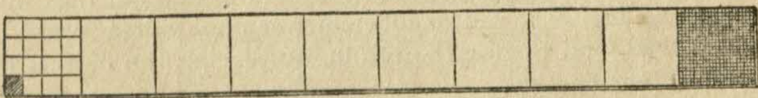


Fig. 188. 1 ACRE = 4 ROODS = 40 SQUARE PERCHES



1 sq. chain =
16 sq. poles
1 acre = 160 sq. poles

Fig. 189

1 sq. chain =
484 sq. yards
1 acre = 4840 sq. yards

An acre or field even in Anglo-Saxon times was a piece of land which had a length of one furlong (furrow's length, or $\frac{1}{8}$ of a mile, or 40 perches) and a breadth of 4 perches. Thus it was a narrow strip of land, the length being ten times the breadth. A square-shaped acre would have sides approximately $69\frac{1}{2}$ yds. long. (See Figs. 187–189.)

Too seldom in schools are the children familiarized with the

size of an acre of land. The use of Gunter's chain, which was invented early in the seventeenth century, is dealt with in full by Watson.

Though the calculation given may be beyond the powers of a child of ten, yet Gunter's chain might with advantage be used to measure out an acre, and for older children such calculations as that included in Watson's book would give a little more interest and life to decimals.

(2) **Exercises in calculating Areas.** The commonest mistake made in teaching area is for the children to be told to construct a rectangle from given data, and then to divide it up into square inches. From a series of such rectangles they deduce the rule: "To find the area of a rectangle multiply the length by the breadth."

A little thought would tell the teacher that this is an altogether unnatural proceeding. Constructing rectangles from given data or of given area should come long after the measuring of existing areas. This measuring of area should be done by superposing some unit of area, and the rule should not be given to the children until they are quite familiar with the idea of area measured by area.

(*a*) *Measuring Given Areas in Square Inches.* It is probably best to start by introducing the square inch. Coloured cardboard square inches are easily obtained, and with them the children should find the surface-area of their exercise-books, pencil-boxes, etc. It will interest them to find out thus who has the largest pocket-handkerchief or piece of cardboard in the class, and to test the results by placing the objects on top of one another.

(*b*) *Building up the Table of Square Measure.* Square feet can be cut out of cardboard or three-ply wood and square yards out of brown paper or American cloth. The connexion between a square foot and a square inch should be found out by the children covering the square foot with square-inch tablets. Before proceeding further they should take their 144 square inches and arrange them to form rectangles of different shapes—*e.g.*, 18 × 8, 24 × 6. It is most important that they should realize that a square foot of material need not form a square. (See Fig. 190 for a simple exercise in

this connexion.) A square yard should then be given to them, and by superposing square feet they will find the connexion between the two units. Various surfaces should then be measured in square feet and square yards—*e.g.*,

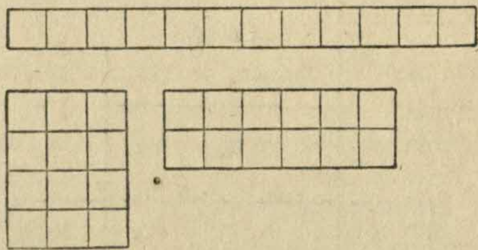


Fig. 190. RECTANGLES MADE WITH TWELVE SQUARE-INCH TABLETS

Questions: "Make as many different-shaped rectangles with these twelve square inches as you can."
"If one square inch represents one square foot, how many square yards are there in each rectangle?"
"If one square inch represents one square chain, mark off an acre."

flower-beds, corridor, class-room, playground. Where possible the square yards should be scratched on the ground or marked out in chalk on the floor.

(*c*) *Calculating Area from Diagrams or Objects made to Scale.* A number of interesting problems on area can be devised with a set of templates. Fig. 191 shows a rectangle in its

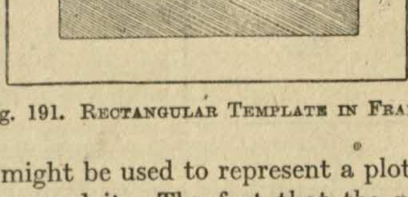


Fig. 191. RECTANGULAR TEMPLATE IN FRAME

frame. This might be used to represent a plot of grass with a gravel path round it. The fact that the main rectangle may be removed makes the working of the calculation clear to a child. (A very good set of templates is published by E. J. Arnold.)

A cardboard box broken down at the corners so that it can be laid quite flat on the table (see Figs. 192 and 193) is an excellent piece of apparatus for calculating area. If the

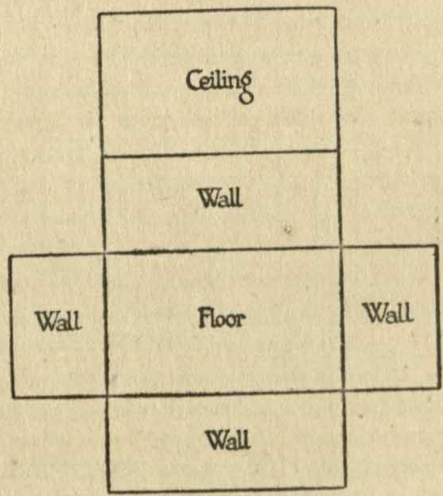


Fig. 192. Box with Sides Broken to show Surface Area of a Room

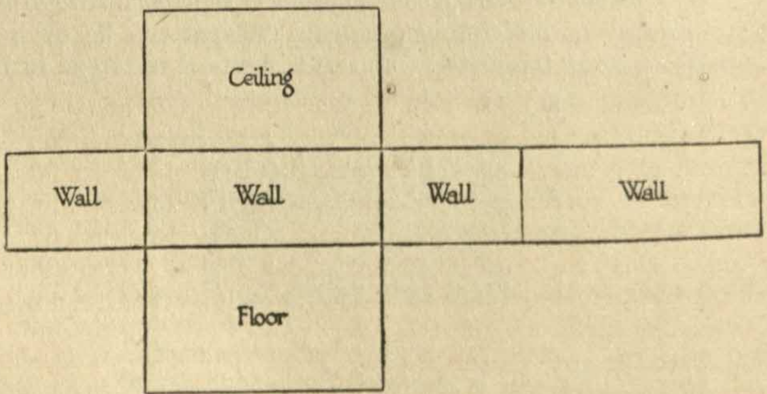


Fig. 193. Alternative Method of Spreading out Box

children have not learnt how to measure to scale the question might be simply, " How much of this pink paper shall I need to line this box ? " If scale has been mastered the box might represent a room, and the area of walls, ceiling, and floor

might be found with a view to calculating the price of papering, painting, and carpeting.

N.B. The subject of scale-drawing and interpretation has purposely been omitted, as it is so admirably dealt with in many elementary geographies.

Section VI. Measurement of Time

(1) **History.** It is not possible to give children under eleven much about the history of our measurement of time.

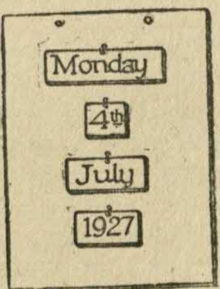


Fig. 194. Wall-calendar in Three-ply Wood

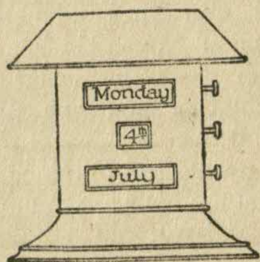


Fig. 195. Small Writing-table Calendar

They should, however, know something of the origin of the names of the days of the week and of the months.

They should also know that the lengths of a day and of a year depend on the movement of the earth round the sun and that of a month on the movement of the moon round the earth. Experience shows that to give small children elaborate lessons on the seasons and day and night is waste of time. Only the most elementary facts should be demonstrated. Fourteen or fifteen is the age at which more advanced work will be appreciated.

(2) **Exercises on Calculation of Time.** (*a*) Very useful pieces of apparatus are shown in Figs. 194 and 195. Every day the children should themselves put out the right tablets or turn the rollers to show the correct day and month. Later on a card-calendar with all the months shown on one sheet provides excellent exercises.

(*b*) From it the children should find out how many days

and weeks there are in a year and in each of the months. Only after this has been done should the following rime be taught:

> Thirty days hath September
> April, June, and November.
> All the rest have thirty-one,
> Excepting February alone,
> Which hath but twenty-eight days clear,
> And twenty-nine in each leap year.

This one-card calendar is also useful in finding intervals between given dates—*e.g.*, "Clare's birthday was on October 21st. To-day is March 5th. How old is she?" "A ship sailed from Bombay on October 15th, and arrived at Southampton on November 8th. How long did the voyage take?"

(*c*) A clock-face is an indispensable piece of apparatus. The children should be told what the minute-divisions stand for, and then be allowed to count them. With slow children good results are obtained by removing one hand. A child can tell the hour approximately by looking at the small hand only, and can tell a lapse of time less than one hour by watching the minute-hand. From the clock-face the child should build up his table of time:

60 minutes = 1 hour.
24 hours = 1 day.

When both hands are taught together, a quarter, a half, and three-quarters of an hour should be landmarks.

(*d*) Both ways of registering the time should be taught —viz., 6.35 or 25 minutes to 7 (both P.M.). Toy watches can be made from very large counters and paper-fasteners or wire. Drill may then take this form:

"Set your watch at 20 minutes to 7; move hands to 8.25. How many minute-divisions has the long hand passed over?"

"When the little hand is exactly at the number XII, where should the big one be?"

"How many minutes in half an hour, a quarter of an hour, three-quarters of an hour?"

"What part of the dial has the big hand passed over in moving from the figure XII to VI?"

(*e*) A.M. and P.M. By letting the children move the hands of their watches or of the clock-face through one day—that is, twenty-four hours, or two complete turns—and then asking them to set them at, for instance, 8 o'clock, the difficulty of knowing in which of the two sets of twelve hours this 8 o'clock is to be counted can be made clear to them.

The signs A.M. (*ante meridiem*, " before noon "—*i.e.*, from midnight to noon) and P.M. (*post meridiem*, " past noon " —*i.e.*, from noon to midnight) must then be given. After this an exercise on their own order of the day will prove both amusing and instructive: " Set your watches at midnight. Now move the hands to your hour of rising, then to breakfast-, school-, and dinner-time, and so on. Each time write down the hour, putting A.M. and P.M. after the figures."

The rime " Rosey Posey," given in Chapter II, affords a good exercise in the use of A.M. and P.M.

(*f*) *Estimating Time.* Children get some idea of time by sitting perfectly still for two minutes, by timing themselves at various tasks, by using an hour-glass (egg-boiling glasses are useful), and by counting seconds. They should be encouraged to time themselves coming to and from school, putting on their boots, packing up at the end of class, learning some piece of poetry or prose by heart. In many schools the timing of lessons, etc., is left entirely to one child. With the introduction of individual work it is hoped that every child will come to realize time and to keep some record of it. Often it is difficult for a teacher to gauge what preparation an individual child is capable of doing at home in one hour. Hence some are overworked and others underworked. If the children took more interest in the distribution of their time in school the teacher would be enabled to apportion the work more fairly. Frequently a child will say, " This has taken me ages," when it has taken only half an hour. Children's ideas of time are usually very vague, and in school they must gradually learn to estimate somewhat more accurately. Of course, no set lessons should be given on this.

CHAPTER X

FRACTIONS

Section I. General Notes on the teaching of Fractions

The idea of fractions should be introduced to the child gradually, and always in connexion with concrete practical work.

Most children become familiar with the signs $\frac{1}{2}$ and $\frac{1}{4}$ in relation to halfpennies and farthings without understanding why these particular signs should be used, and this is perhaps the best way of introducing children of six years of age to fractions. In their counting exercises they will halve and double numbers, and in weighing and measuring they will come across the $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ in other connexions. Hence at eight years of age they will have some very definite ideas about fractions, and will be ready for more formal lessons. Then all the foregoing work will have to be revised and extended.

The words 'numerator' and 'denominator' should be taught; to postpone their use and to substitute 'naming figure,' 'numbering figure,' seems to be a great mistake, for most children of eight like the sound of the longer and less familiar words, and do not find them confusing.

A child of eight or eight and a half may, therefore, be taught that "The denominator is the lower figure of a fraction, and it tells us into how many parts the whole has been divided," and that "The numerator is the upper figure of a fraction, and it tells us how many parts have been taken."

Thus the sign $\frac{1}{4}d.$ tells us that one penny has been divided into four parts, and one of these parts has been taken.

The children must be given constant practice in representing fractions with concrete material until they are perfectly clear as to the meaning of the above terms.

For this purpose they should first be taught with some accurate apparatus such as is described in the next section. After this a graded set of exercises should be given so as to introduce fractions in relation to money, weight, etc. The course would therefore include:

(1) Formal exercises on the properties of fractions, using the method of dividing wholes as explained in Section II.
(2) Dividing a number of discrete things into groups (see partition, Chapter VII).
(3) Calculations in money.
(4) Measurements of weight, capacity, area, etc.
(5) Calculations in time (hours, half-hours, etc.).

The fundamental ideas which must be made quite clear are best classified under three headings:

(1) **Facts which may be deduced directly from the Definitions of Numerator and Denominator.** (*a*) *The larger the denominator the smaller will the parts be,* and hence the larger the denominator the greater the number of parts that must be taken to build up one whole. At first the child should be required to arrange in ascending or descending order fractions having unity as numerators, thus:

$$\tfrac{1}{9},\ \tfrac{1}{7},\ \tfrac{1}{6},\ \tfrac{1}{4},\ \tfrac{1}{3},\ \tfrac{1}{2}.$$

Later this exercise may be extended to fractions with numerators greater than unity.

(*b*) *A fraction is increased when*

(i) The *numerator* is multiplied by a whole number, for we thus increase the number of parts taken.
(ii) The *denominator* is divided by a whole number, for we thus increase the size of each part taken.

(*c*) *A fraction is decreased when*

(i) The *numerator* is divided by a whole number.
(ii) The *denominator* is multiplied by a whole number.

Children should be given practice in doubling, trebling, halving, etc., quite simple fractions.

(2) **The Equivalence of Fractions.** The children must be taught to realize that, for instance, $\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$, and so

N

on—in other words, the fact that if both denominator and numerator are multiplied or divided by the same number the fraction remains unchanged in value. This is a very important point, for on it are based addition, subtraction, and division of fractions.

(3) **A Quantity multiplied by a Fraction results in a Diminished Quantity.** This is always a difficulty with small children. The best plan is to *use the word 'of' and not the multiplication sign.* Such facts as $\frac{1}{2}$ of $\frac{1}{2} = \frac{1}{4}$ and $\frac{1}{3}$ of $\frac{1}{2} = \frac{1}{6}$ are very easy to demonstrate, and the child will experience no difficulty whatever. It is only later on that the meaning up to now attached to multiplication in the child's mind—viz., that multiplication is continued addition—need be extended. (See *The Psychology of Number*, p. 258.)

Obviously the above fundamental ideas cannot be put into words by a small child, but the teacher must see that exercises are given that secure the understanding of them.

Section II. Apparatus

The applications of fractions are so numerous that there is bound to be a certain difference of opinion as to the best apparatus for the first formal work.

Two kinds are recommended :

(1) Apparatus based on measurement of length.

(2) Apparatus based on sections of a circle, square, or rectangle.

(1) **Apparatus based on Measurement of Length.** McLellan and Dewey say:

> As in "integers," so in fractions the idea and process of measurement should be ever present. To begin the teaching of fractions with vague and undefined "units" obtained by breaking up equally undefined wholes—the apple, the orange, the piece of paper, the pie—may be justly termed an irrational procedure. Half a pie, for example, is not a numerical expression at all, unless the pie is defined by weight or volume ; the constituent factors of a fraction are not present ; the unity of arithmetic is ignored ; the process of fractions is assumed to be something different from that of measurement. . . .

The primary step in the explicit teaching of fractions—that is, in making the habit of fractioning already formed an object of analytical attention—is to make perfectly definite the child's acquaintance with certain standard measures, their sub-divisions, and relations. In all fractions—because in all exact measurement—there must be a definite unit of measure.[1]

Cutting up apples and cakes is allowable in a conversation lesson in the baby-room, but not in a formal lesson on fractions. A foot-rule and inch-pieces seem the best apparatus

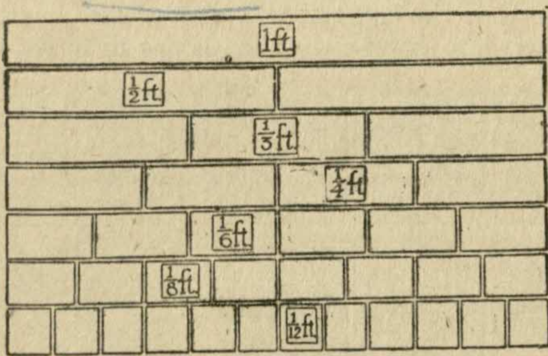


Fig. 196

for introducing formal work in fractions for the following reasons:

(a) Measurement of length is a simple operation.

(b) Twelve is a good number for factorizing, for we can get $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{12}$ of it as whole numbers.

(c) The pieces are large.

(d) The results are concrete—*e.g.*, $\frac{1}{4}$ foot = 3 inches.

(e) The exercises are of the utmost practical value, not only in calculations involving measuring, but also in money sums.

The pieces contained in the foot-box, the use of which is strongly recommended, are: Three pieces 1 ft. long; 2 of 6 in.; 3 of 4 in.; 6 of 2 in.; 8 of $1\frac{1}{2}$ in.; 12 of 1 in.; 1 each of 5 in., 7 in., 8 in., 9 in., 10 in., 11 in. They are graduated in inches on one side only. (See Fig. 196.)

[1] *The Psychology of Number.*

It is an excellent exercise for children of nine or ten to cut strips of cardboard and make all the pieces in the foot-box for use in the lower classes.

(2) **Apparatus based on Sections of a Circle, Square, or Rectangle.** The practice of teaching the value of fractions by distributing pieces of paper and allowing the children to fold, refold, and then cut has certain dangers. First of all, the papers given out are not always the same size. Hence Lucy wonders why her piece marked $\frac{1}{2}$ is smaller than Mary's. Secondly, when the half has been divided into quarters and the quarters into eighths the child is apt to forget what he

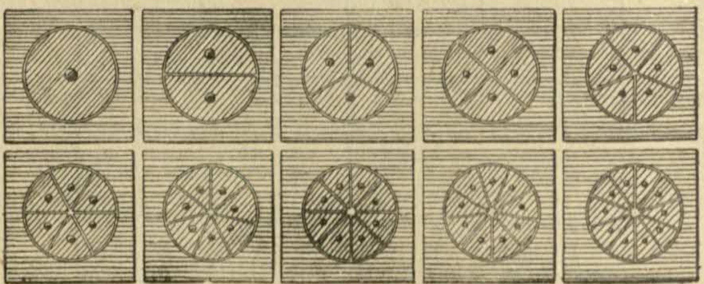


Fig. 197. Montessori Fraction Apparatus
By permission of Messrs Philip and Tacey, Ltd.

has done and what he started with. Thirdly, for small children folding and cutting are difficult processes in themselves, and hence are often inaccurately done. Such work is most useful when the child is eight or nine. Then he should be given four or five pieces of paper exactly the same size, and one of these he should not cut up, but keep entire. *On this whole* he can then build up equal wholes by placing the cut pieces together.

For the first exercises it is better to provide readymade and accurately cut circles or squares. When this is done the children will almost invariably cut or draw similar ones for themselves, and a standard of accuracy will have been given to them. Their results will be more likely to be exact, as they will realize what they are doing and why.

(*a*) *Dr Montessori's apparatus* is shown in Fig. 197, and is

described under the heading "Geometry" in vol. ii of her *Advanced Montessori Method.* This apparatus is very perfect, and every school should possess one set if possible. Not only is it valuable for geometrical work in the upper classes, but even little children of seven can make quite complicated calculations with it. Though at this age they do not fully understand, for instance, converting fractions to a common denominator, yet working with such an accurate piece of apparatus prepares them for understanding the process later.

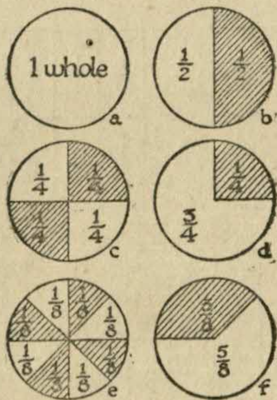


Fig. 198. "PROGRESS" GAME No. 10

The only drawbacks to it are that it is expensive (£2 12*s.*) and takes up rather much room.

On account of the handles a small child is apt to think that the circle having ten handles is more important than those with only one or two! The handles also interfere with the equivalence of the pieces being proved by placing them one on the top of the other, which method is satisfying to a child.

(*b*) *Cardboard Circles.* The idea of divided circles has been used in the "Progress" Games 10 to 13. The pieces to be cut are accurately marked out, and being in cardboard are inexpensive, as well as fairly durable. Each side is of a different colour, so that, by turning one piece, the significance of a sum may be clearly shown. Fig. 198 shows the pieces in Game 10. Fig. 199 shows similar apparatus based on the

4-in. square. With it an improper fraction may be represented as a rectangle.

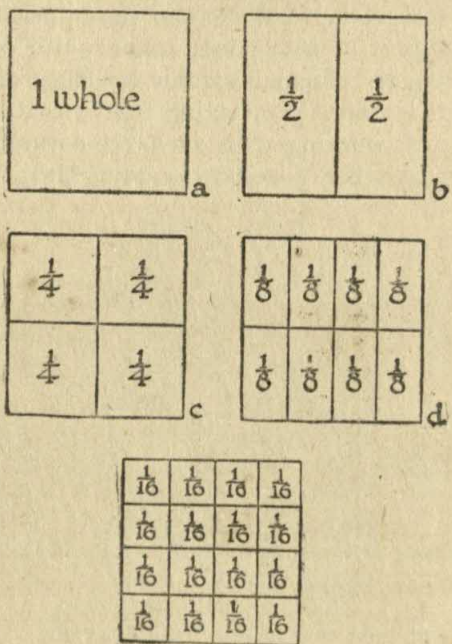


Fig. 199

Section III. Exercises with the Foot-box

The important point to bear in mind is that these exercises are useful in so far as they inculcate the necessary fundamental ideas of fractional quantities, and that when once these ideas are clearly grasped there is no need to continue practical exercises except as occasional tests. The fundamental ideas that must be made quite clear are given in Section I.

The order of these exercises should be changed according to the previous knowledge and experience of the child. They are merely intended to be suggestive.

First Exercise. The child builds up the foot or 'whole,'

using like pieces each time. A little ticket with the fraction sign is put on each piece. The results recorded would be:

$$\tfrac{1}{2} + \tfrac{1}{2} = \tfrac{2}{2} = 1.$$
$$\tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{3}{3} = 1.$$

And so on with fourths, sixths, eighths, and twelfths.

Second Exercise. The child now proceeds to find out the exact value of each piece in inches. The pieces are turned to the graduated side. The results might be recorded as 6 in. $\times$ 2 = 12 in. = 1 ft. (see Fig. 200), and as $\frac{1}{2}$ ft. = 6 in. (Fig. 201), and as 6 in. = $\frac{1}{2}$ ft. (Fig. 202).

Card 1
$\boxed{6}$ in. × $\boxed{2}$ = 1 ft.
$\boxed{4}$ in. × = 1 ft.
in. × = 1 ft.
in. × = 1 ft.
in. × = 1 ft.
in. × = 1 ft.

Fig. 200

Card 2
$\frac{1}{2}$ ft. = $\boxed{6}$ in.
$\frac{1}{4}$ ft. = in.
$\frac{1}{3}$ ft. = in.
$\frac{1}{6}$ ft. = in.
$\frac{3}{4}$ ft. = in.
$\frac{1}{12}$ ft. = in.

Fig. 201

Card 3
3 in. = $\boxed{\frac{1}{4}}$ ft.
6 in. = ft.
2 in. = ft.
4 in. = ft.
9 in. = ft.
8 in. = ft.

Fig. 202

Third Exercise. A value is now found for each of the remaining pieces. This is done by measuring them with the known pieces—*e.g.*, the 9-in. piece is measured with the 3-in. or $\frac{1}{4}$-ft. piece, and is then marked "$\frac{3}{4}$ ft."; and "9 in. = $\frac{3}{4}$ ft." is recorded. So also the 8-in. piece is measured with the 4-in. or $\frac{1}{3}$ ft. piece, and "$\frac{1}{3}$ ft. + $\frac{1}{3}$ ft. = $\frac{2}{3}$ ft. = 8 in." is recorded.

Fourth Exercise. As many equivalent fractions as possible are found by measuring the pieces with each other—*e.g.*,

$$\tfrac{1}{4}\text{ ft.} + \tfrac{1}{4}\text{ ft.} = \tfrac{2}{4}\text{ ft.} = \tfrac{1}{2}\text{ ft.} \qquad \tfrac{6}{12}\text{ ft.} = \tfrac{3}{6}\text{ ft.} = \tfrac{1}{2}\text{ ft.}$$

and so on.

The teacher must be on the look-out for the moment when a little suggestion will lead the child to formulate the fundamental idea given above—viz., that of the equivalence of fractions. This moment may not come until the child has

worked through several more types of exercises, and on no account should the rule be given before the child has had some experience in fractions. Once the rule has been formulated he should be encouraged to experiment and test his own answers—*e.g.*, " $\frac{1}{2} = \frac{3}{6} = \frac{9}{18}$. Is this true ? " This type of exercise will have to be worked on squared paper if the pieces in the foot-box are insufficient.

Fifth Exercise. Lengths exceeding one foot may be formed by several rods being placed end to end. Each result should be labelled : $1\frac{1}{2}$ ft., $1\frac{1}{4}$ ft., $1\frac{1}{3}$ ft., $\frac{5}{6}$ ft., $\frac{5}{8}$ ft., $\frac{7}{12}$ ft., $\frac{5}{12}$ ft., $\frac{3}{8}$ ft.

Card 4
$\frac{1}{2}$ + $=1$
$\frac{1}{4}$ + $=1$
$\frac{3}{4}$ + $=1$
$\frac{2}{3}$ + $=1$
$\frac{1}{6}$ + $=1$
$\frac{1}{3}$ + $=1$

Fig. 203

Card 5
$1\frac{1}{2} = \frac{\boxed{5}}{4}$
$2\frac{1}{4} = \frac{\quad}{4}$
$\frac{3}{4} = \frac{\quad}{4}$
$\frac{4}{8} = \frac{\quad}{4}$
$\frac{8}{12} = \frac{\quad}{4}$
$1\frac{3}{4} = \frac{\quad}{4}$

Fig. 204

Questions such as the following might be given :

" How many halves, fourths, sixths, eighths, twelfths, are there in $1\frac{1}{2}$ ft. ? "

" How many sixths, twelfths, thirds, in 1 ft. ? "

All these can be worked out with the rods.

Result-cards similar to those shown in Figs. 203 and 204 are useful for individual work on the above exercises.

Before proceeding much practice should be given in :

(1) Adding fractions with the same denominators and finding the number of wholes—*e.g.*, $\frac{3}{8}$ ft., $\frac{2}{8}$ ft., $\frac{5}{8}$ ft., $\frac{4}{8}$ ft., $\frac{1}{8}$ ft.

(2) Completing wholes—*e.g.*, " What must I add to $\frac{3}{4}$ ft. to make 1 ft. ? " " How many thirds must be added together to make 2 ft. ? "

(3) Breaking up wholes—*e.g.*, " How many quarters in $1\frac{1}{4}$? "

And so on.

Sixth Exercise. Values should now be found for such expressions as the following:

$\frac{3}{4}$ ft. + $\frac{1}{2}$ ft.	$1\frac{1}{2}$ ft. + $1\frac{1}{4}$ ft.	$\frac{1}{2}$ ft. + $\frac{1}{4}$ ft.	$\frac{3}{8}$ ft. + $\frac{1}{4}$ ft.
$\frac{5}{8}$ ft. − $\frac{1}{2}$ ft.	$\frac{1}{4}$ ft. + $\frac{1}{2}$ ft.	$\frac{1}{8}$ ft. + $\frac{1}{2}$ ft.	$\frac{3}{8}$ ft. + $\frac{1}{4}$ ft.
$\frac{5}{8}$ ft. + $\frac{3}{4}$ ft.	$\frac{5}{12}$ ft. + $\frac{1}{2}$ ft.	$\frac{3}{4}$ ft. − $\frac{3}{8}$ ft.	$\frac{1}{4}$ ft. − $\frac{1}{8}$ ft.

The sums might be worked out on paper, and the child led to formulate the rule for adding and subtracting fractions with unlike denominators. In all the above examples the common denominator is given.

Seventh Exercise. The same as the sixth, except that examples are given in which new denominators have to be found by measurement—*e.g.*, $\frac{2}{3}$ ft. + $\frac{1}{4}$ ft.

To add these the child will discover that he must measure each with the $\frac{1}{12}$ piece. Too much time should not be spent over this, for from the foregoing work he will already know that $\frac{2}{3}$ ft. = $\frac{8}{12}$ ft. and $\frac{1}{4}$ ft. = $\frac{3}{12}$ ft.

The second example might be $\frac{2}{3}$ ft. + $\frac{1}{2}$ ft., as this will impress the fact that the largest possible piece should be taken to measure with—in this case not the twelfth of a foot, but the sixth.

No examples more difficult than $\frac{1}{3} + \frac{3}{8}$, $\frac{2}{3} + \frac{1}{8}$, $\frac{3}{4} + \frac{1}{8}$ should be attempted until the method for finding L.C.M. is taught. Then drill should follow, without the examples being tested with the rods or even being fully worked out —*e.g.*, "What number must be taken as common denominator when the following are to be added: (1) halves and thirds; (2) thirds and quarters; (3) eighths and thirds?" and so on.

Eighth Exercise. Further examples in addition and subtraction, involving whole numbers—*e.g.*, $3\frac{3}{4} - 2\frac{1}{2}$; $4\frac{1}{6} - 2\frac{5}{6}$.

The fractions should be dealt with first, and whatever method of subtraction has been adopted in earlier work should be adhered to here.

If *decomposition* has been taught the sum $4\frac{1}{6} - 2\frac{5}{6}$ will be worked as follows:

$\frac{5}{6}$ cannot be taken from $\frac{1}{6}$, therefore take 1 from the 4 units; then $\frac{5}{6}$ from $\frac{7}{6} = \frac{2}{6}$.

Then in the units $3 - 2 = 1$.

Thus the sum has really been treated as $3\frac{7}{8} - 2\frac{5}{8}$.

In *equal additions* or *complementary addition* $\frac{8}{8}$ is added to the fractional part of the minuend and 1 to the units of the subtrahend. This sum is really treated as $4\frac{7}{8} - 3\frac{5}{8}$.

See Chapter V for details of these methods.

Ninth Exercise. The fact that a fraction multiplied by a fraction results in a diminished quantity must now be brought

$\frac{3}{4} \div \frac{1}{8}$	$= \frac{6}{8} \div \frac{1}{8}$	= 6 eighths ÷ 1 eighth	= 6
$\frac{3}{2} \div \frac{1}{4}$	$= \frac{6}{4} \div \frac{1}{4}$	= 6 quarters ÷ 1 quarter	= 6
$\frac{5}{2} \div \frac{5}{4}$	$= \frac{10}{4} \div \frac{5}{4}$	= 10 quarters ÷ 5 quarters	= 2
$\frac{3}{4} \div \frac{3}{8}$	$= \frac{6}{8} \div \frac{3}{8}$	= 6 eighths ÷ 3 eighths	= 2
$\frac{3}{4} \div \frac{1}{2}$	$= \frac{3}{4} \div \frac{2}{4}$	= 3 quarters ÷ 2 quarters	$= 1\frac{1}{2}$
$\frac{5}{8} \div \frac{1}{4}$	$= \frac{5}{8} \div \frac{2}{8}$	= 5 eighths ÷ 2 eighths	$= 2\frac{1}{2}$
$\frac{5}{4} \div \frac{5}{8}$	$= \frac{10}{8} \div \frac{5}{8}$	= 10 eighths ÷ 5 eighths	= 2
$\frac{10}{3} \div \frac{5}{6}$	$= \frac{20}{6} \div \frac{5}{6}$	= 20 sixths ÷ 5 sixths	= 4

Fig. 205. Table from which to deduce the Rule for Division of Fractions

home to the child. This can be done by saying, "What is $\frac{1}{2}$ of $\frac{1}{2}$ ft.; $\frac{1}{3}$ of $\frac{1}{2}$ ft.; $\frac{1}{2}$ of $\frac{1}{8}$ ft.; $\frac{1}{2}$ of $\frac{3}{4}$ ft.; $\frac{1}{2}$ of $\frac{5}{8}$ ft.; $\frac{1}{3}$ of $\frac{1}{4}$ ft.?" and so on. All these examples may be worked with the rods.

Tenth Exercise. It will be noticed that the above exercises have introduced the child to addition, subtraction, and multiplication of fractions, but that division of a fraction by a fraction remains to be taught. This is, perhaps, best done by working and tabulating a series of very easy examples which can all be worked out practically with the rods or on squared paper.

Such a simple table is given in Fig. 205, and from a study of it the child should be able to formulate the rule, "To divide a fraction by a fraction, invert the divisor and multiply."

The following is another explanation of division of a fraction

by a fraction which is sometimes given, but is less satisfactory than the above.

Suppose the sum is $\frac{9}{4} \div \frac{3}{8}$. Divide $\frac{9}{4}$ by 3. This gives $\frac{3}{4}$, but this is too small an answer, for $\frac{3}{8}$ will go into $\frac{9}{4}$ eight times oftener than 3 will, for it is eight times smaller; hence we must multiply by 8. Thus we have inverted the divisor and multiplied.

SECTION IV. EXERCISES WITH DIVIDED CIRCLES. PARTITION

(1) "Progress" Game 10 (Fig. 198) consists of circles blue on one side and buff on the other. Some of the circles have lines dividing them into halves, quarters, and eighths. They should be cut up very accurately along these lines. The following are some suggestive exercises:

(*a*) Make up three circles. Each circle should contain pieces of one size only. Results: $\frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$. And similarly with quarters and eighths.

(*b*) Measure with a $\frac{1}{4}$ piece to find how many quarters there are in $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, etc.

(*c*) Measure all the pieces with the $\frac{1}{8}$ pieces.

(*d*) Find the value of all the pieces not used in (1).

(*e*) How much is $\frac{3}{4} + \frac{1}{2}$? $1\frac{1}{2} + \frac{5}{8}$?

(*f*) By turning some of the pieces the teacher can set sums of this type: How much of the circle is blue? How much is buff? Subtract the blue from the buff. Add the blues in these three circles and subtract the result from the sum of the buffs. And so on. Children love setting themselves such sums.

Simple problems might then be given which could be solved by placing sections of circles together—*e.g.*, "A cake is to be divided between Jim, Jack, and Baby. Jim gets $\frac{1}{2}$; Baby gets $\frac{1}{8}$. How much is left for Jack?"

The child should be allowed to experiment himself with the pieces, and should be encouraged to write down all his results in equation form.

Fig. 206 shows how he may be led to see that $\frac{3}{4}$ of 1 is the

same as $\frac{1}{4}$ of 3. This fact is very important, for he will use it to express remainders (see Chapter VII). He should understand that $\frac{3}{4}$ may be used to represent three wholes divided by four.

"Progress" Games 11, 12, and 13 deal with more difficult fractions.

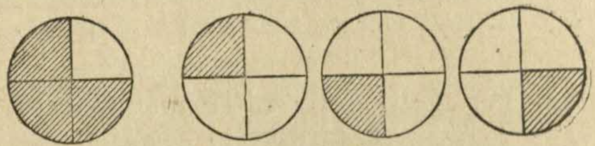


Fig. 206. To show that $\frac{3}{4}$ of $1 = \frac{1}{4}$ of 3

(*b*) *Dividing a Number of Discrete Objects into Groups.* This may be dealt with as in Chapter VII. For instance, if twelve cards are dealt out to four people, each will get 3 or $\frac{1}{4}$ of the number dealt out. After some experimenting the child should be given examples such as the following:

$$\tfrac{1}{2} \text{ of } 18 = \text{——}. \qquad 3 = \text{——} \text{ of } 9 = \text{——} \text{ of } 12.$$

Given that $9 \times 4 = 36$, and using only these three numbers, write down as many equations as you can. Results:

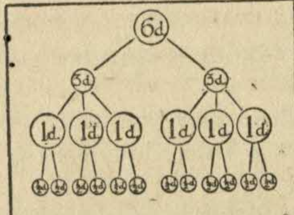
$$\begin{aligned} \tfrac{1}{4} \text{ of } 36 &= 9. \\ \tfrac{1}{9} \text{ of } 36 &= 4. \\ 36 \div 9 &= 4. \\ 36 \div 4 &= 9. \\ 4 \times 9 &= 36. \end{aligned}$$

Section V. Application of Fractions to Money and to Weights and Measures

(1) **Application to Money.** (*a*) Many of the exercises given in Chapter VIII might be used to teach fractions. For instance, by the use of the money chart the child can find the value of $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{3}{4}$, $\frac{1}{8}$, $\frac{3}{8}$, $\frac{2}{3}$, etc., of one shilling. He may then be told to put twelve pennies with 'heads' up on to the chart, and to turn $\frac{2}{3}$, $\frac{3}{4}$, of them, and so on.

(*b*) By putting sixteen shillings and eight sixpenny-pieces on the chart the value of 2*s*. 6*d*., 2*s*., 5*s*., 10*s*., 15*s*., 12*s*. 6*d*., 7*s*. 6*d*., and so on as fractions of £1 may be found.

(*c*) To find £$\frac{1}{3}$, twenty shilling pieces are taken and dealt out. There will be three piles of 6*s*. The remaining two shillings are then changed for twenty-four pennies, and these



$\frac{1}{2}$ of 6d.=
$\frac{1}{3}$ of 6d.=
$\frac{1}{4}$ of 6d.=
$\frac{1}{12}$ of 6d.=
$\frac{1}{2}$ of 3d.=

Fig. 207

$\frac{1}{3}$s.=
$\frac{2}{3}$s.=
$\frac{1}{4}$s.=
$\frac{3}{4}$s.=

Fig. 208

too are dealt out (see Chapter VIII). From the result, £$\frac{1}{3}$ = 6*s*. 8*d*., the values of £$\frac{2}{3}$, £$\frac{1}{6}$, £$\frac{5}{6}$ are then found.

Figs. 207, 208, and 209 show simple exercise-cards.

Fig. 210 shows a chart of the aliquot parts of a pound. These are very important, and they must ultimately be

6d.= of 1/6
6d.= of 2/-
6d.= of 1/-
6d.= of 3/-
6d.= of 5/-

Fig. 209

<table>
<tr><td colspan="8">£1</td><td>whole</td></tr>
<tr><td colspan="4">10/-</td><td colspan="4">10/-</td><td>halves</td></tr>
<tr><td colspan="2">5/-</td><td colspan="2">5/-</td><td colspan="2">5/-</td><td colspan="2">5/-</td><td>quarters</td></tr>
<tr><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>2/6</td><td>eighths</td></tr>
</table>

<table>
<tr><td colspan="4">6/8</td><td colspan="4">6/8</td><td colspan="4">6/8</td><td>thirds</td></tr>
<tr><td colspan="2">3/4</td><td colspan="2">3/4</td><td colspan="2">3/4</td><td colspan="2">3/4</td><td colspan="2">3/4</td><td colspan="2">3/4</td><td>sixths</td></tr>
<tr><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>1/8</td><td>twelfths</td></tr>
</table>

Fig. 210

memorized by every child. The less important fractional parts can readily be deduced from these.

(2) **Application to Length, Area, Weight, etc.** Exercises in fractions should include very simple examples in measuring length and area, in weighing out quantities, and in finding capacities in pints, etc.

Suggestions for exercises will be found in the chapter on money calculations, more especially in the section on shopping, and in Chapter IX on weights and measures.

(*a*) The following are a few suggestive exercises involving drawing:

(i) Draw a line $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$, $1\frac{1}{2}$, and 2 times as long as a given line.

(ii) If this line measures 3 in., draw one measuring 4 in., $1\frac{1}{2}$ in., $\frac{1}{2}$ in. Do not use a ruler, but a strip of paper.

(iii) If this line measures 2 units, draw lines to represent $\frac{1}{2}$, $\frac{3}{4}$, $1\frac{1}{2}$, 3 units.

(iv) Draw a line half the length of your penholder.

(v) Draw a line 3 in. long. Mark off $\frac{1}{3} + \frac{1}{6}$ of the line.

(vi) Draw a line 4 in. long. Mark off $\frac{1}{2} - \frac{1}{4}$ of the line.

(vii) Draw a line 6 in. long. Mark off $\frac{1}{4}$ of $\frac{1}{3}$ of the line.

(viii) Draw a line 5 in. long. Mark off and divide by 2.

(ix) Show by drawing a straight line and dividing it that $\frac{1}{2}$ of $1\frac{1}{3} = \frac{2}{3}$.

Exercises involving the Drawing of Rectangles.

(i) Draw a rectangle 4 in. by 3 in. and mark off $\frac{1}{4}$ of it.

(ii) Draw a rectangle 4 in. by 3 in. and mark off $\frac{1}{4} + \frac{1}{6}$ of it.

(iii) Draw a rectangle 4 in. by 3 in. and mark off $\frac{1}{2}$ of $\frac{1}{3}$ of it.

(iv) Draw a rectangle 4 in. by 3 in. and mark off $\frac{1}{4} \div 3$ of it.

(v) Show by drawing rectangles the value of $\frac{1}{2}$ of $1\frac{1}{3}$.

(vi) Show by drawing rectangles that $\frac{2}{3}$ of 1 is the same as $\frac{1}{3}$ of 2, and that $\frac{3}{4}$ of 1 is the same as $\frac{1}{4}$ of 3.

This last example is extremely important, as it gives a new meaning to fractions. This meaning should also be shown with the foot-box and with the "Progress" circles (see Fig. 206).

(vii) If a given rectangle shows a third of the top of a box, draw two diagrams showing possible shapes of the box.

(viii) Cut each of three squares of paper into quarters so that you get three sets of quarters of different shapes.

(ix) Show by a diagram that $\frac{1}{4}$ of $\frac{1}{3} = \frac{1}{12}$.

Many of the diagrams given in books are too complicated to be of use in explaining fractions, but they have a value in the work required to build them up. For children under twelve only the simplest exercises should be given. It is extremely important that the teacher should realize which of the many diagrams recommended in arithmetic should be done accurately and which may be merely rough sketches. Much time is wasted by teachers insisting on accurate diagrams when a few strokes of the pen would suffice to show the 'situation,' 'character,' or relative positions of the components of the problem. A habit of rapidly sketching a problem is to be cultivated, as it elucidates the situation, exposes ambiguities, etc. Such a habit is killed if the teacher insists on all diagrams being accurately to scale. At the same time, it is essential that the children be able to represent a quantity accurately in a graph, and indeed, in the above exercises, *where a representation of a quantity* is asked for, accuracy is very important.

(*b*) With regard to *measurement of weight and capacity*, the great thing to make sure of is that the child realizes that if he knows that 1 lb. equals 16 oz., he also knows the value of $\frac{1}{2}$ lb., $\frac{1}{4}$ lb., etc., in ounces, and *vice versa*—that knowing 8 oz. = $\frac{1}{2}$ lb. he can express any number of ounces as fractions of a pound.

(*c*) *Measurement of Time.* A clock-face with movable hands should be used to work such examples as the following :

(i) Place the little hand pointing to XII and the big hand to I. What part of the whole circle lies between them ?

(ii) What part of the whole circle is marked out by ten of the minute-divisions ? By twenty, fifteen, thirty, seven and a half ?

(iii) Fill in the gaps in the following statements :

One complete circle has —— minute-divisions.
15 minute-divisions equal —— of a whole circle.
5 minute-divisions equal —— of a whole circle.
$\frac{1}{2}$ whole round equals —— twelfths.
$\frac{1}{3}$ whole round equals —— twelfths.

SECTION VI. COMMON FAULTS

Before closing this chapter it may be well to point out some of the more common mistakes made by children when working with fractions.

(1) They are apt to cancel clumsily. This comes from their using too small factors and placing the figures vertically above one another. Though it is not advisable to make a child find the G.C.M. of numerator and denominator by an elaborate method, yet he should be trained to factorize rapidly. He should be made to realize the unwieldiness of cancelling as above described, and to look to see if he can find a higher factor to use. Some teachers find it a good plan to insist that the fraction be written out horizontally after two cancellings, but in examples containing several fractions this would be waste of time. All cancelling should be checked either by attempting to get back the original fraction by 'multiplying up' again or by 'multiplying across.' The drawback of the latter method is that it is long in the case of fractions containing big numbers.

(2) Some children cancel when there is a plus or minus sign between the fractions, or they

(3) Add or subtract equal amounts from numerator and denominator.

Not only should the absurdity of these proceedings be demonstrated to them concretely or graphically, but they should be made to prove their own mistakes. When the teacher is quite sure that their foundations are all right a few definite working rules should be given.

(4) When long, complex fractions are given and children are taught to deal first with the numerator and then with the denominator there is a danger of part of the fraction being 'left behind' and overlooked. The objection made by teachers to working such examples in 'one piece' is that it may require more steps and imply writing more figures, but it is possible to deal with the numerator first and then put the denominator in *without* separating them completely.

CHAPTER XI

DECIMALS AND THE METRIC SYSTEM

SECTION I. WHAT TO TEACH

IN the following passages the order in which vulgar and decimal fractions may be taken is discussed:

> It is the traditional practice to study vulgar fractions before decimals. To a certain extent this plan is sound, for 'halves' and 'quarters' are easier than the easiest decimals, but once the beginner can deal with these very elementary vulgar fractions, there is no reason why he should not proceed to decimals. There is indeed every reason why he should study decimals before occupying himself with vulgar fractions of a difficult kind. Vulgar fractions with large denominators are cumbersome and of limited utility, whilst decimals are comparatively easy to handle, and have many practical applications.[1]

> The question of sequence of common and decimal fractions is one which has recently been much discussed. It is easy to dismiss the whole subject by some such remark as, "logically the decimal fraction comes first, because it grows naturally out of our number system," and this is frequently done in some educational sheets. . . . But just as strong an argument can be advanced by saying that psychologically the common fraction should precede, because the concept is the simpler, that historically it was in use long before the decimal system of writing numbers was known, to say nothing of the decimal fraction. . . .

> The question is really, however, one belonging rather to the old-fashioned course than to the modern, to the days when the pupil was expected to 'master' common fractions before studying the decimal. Our modern arithmetics of any standing follow no such plan. The fact is, no one ever thinks, practically, of teaching $0{\cdot}5$ before $\frac{1}{2}$, or $0{\cdot}25$ before $\frac{1}{4}$. The simple fractions $\frac{1}{2}$ and $\frac{1}{4}$ enter into the work of the first year; the forms $0{\cdot}5$, $0{\cdot}25$, represent a much greater degree of abstraction, and hence should have place considerably later.

> But on the other hand, as between adding $0{\cdot}5$ and $0{\cdot}25$ or $\frac{127}{349}$ and $\frac{827}{941}$, there can be no question as to which should have the first place. And hence the conclusion will probably be

[1] *Suggestions for the Consideration of Teachers*, p. 192.

reached by most teachers that the elementary treatment of simple fractions has the first place, but that, long before the pupil comes to the serious difficulties of the common fraction, the tables of United States money, or possibly those of the metric system, should make him familiar with the decimal forms and the simple operations therewith.[1]

It is not desirable that children of this age should be introduced to all the tables of metric weights and measures. It is true that in schools where metric balances are supplied the children might well learn the gramme and kilogramme; otherwise these should be postponed. Neither should any other measures be taught beyond those given below, except perhaps the kilometre.

It will not be necessary to go into details of the four rules, because it is considered that the following only should be taught to a child under ten:

(1) Very simple addition and subtraction sums involving not more than two or three decimal places.
(2) Multiplication and division of a decimal by 10, 100, 1000.
(3) Very simple multiplication and division by a whole number—*e.g.*, 21·2 ÷ 4.

Section II. First Steps in Decimals

(1) **Exercises on the Inch and Tenth of an Inch.** Rulers divided into inches and tenths of an inch should be provided. The children are then taught how to measure lines and objects and to record results in decimal notation: 1·1 in., 2·2 in., 1·5 in., 1·7 in., and so on.

They will readily understand that the figure to the right of the decimal point in any of the examples given is ten times smaller than that to the left. They will realize what one-tenth of an inch means, and thus the way will have been prepared for introducing them to the metric system.

They should draw lines of given length; produce or add to them by definite lengths—*e.g.*, 3·5 in. + 1·2 in.; cut off or subtract parts of them—*e.g.*, 3·5 in. – 2·2 in. Exercises should also be worked on paper ruled in inches and tenths.

[1] Smith, *Teaching of Elementary Mathematics* (Macmillan Co., New York).

(2) **The Metre, Decimetre, Centimetre.** As soon as the children have grasped the fact that, for instance, three-tenths of an inch may be represented as ·3 they should be introduced to metric units of length. Every child should have a strip of paper (ribbon paper does very well) one metre long; also

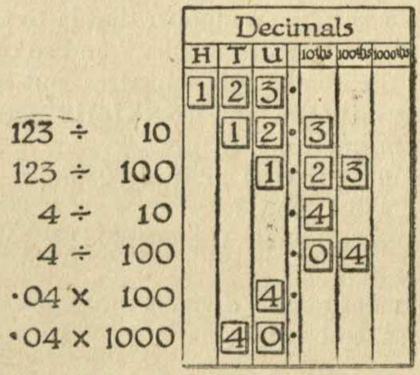


Fig. 211. EXERCISE WORKED WITH DIGIT-TABLETS ON A DECIMAL-BOARD

pieces one decimetre and one centimetre long. With these they should be required to build up the table:

10 centimetres = 1 decimetre.
10 decimetres = 1 metre.

By means of squared paper the metre strips of paper should be graduated and then used for making actual measurements. The following facts should be made quite clear:

10 decimetres = 1 metre.
1 decimetre = ·1 metre.
100 centimetres = 1 metre.
1 centimetre = ·01 metre.

Millimetres might be left until later, but if introduced here the child should not be required to work with ruler or paper graduated in millimetres. These are much too fine to be good for a child's eyes. He should be taught to 'imagine' the centimetre divided into ten parts. He could, for instance, be required to rule lines approximately 2·5 cm. long, etc.

(3) **Exercises with Decimal-board.** The decimal-board shown in Fig. 211 is useful for teaching multiplication and division

by ten or by a multiple of ten. It is important that at first the child should look upon the decimal point as fixed, and should move the figures. The following exercises are suggestive:

"Put digit-tablets on your board to represent 123."
"Move them so as to get a number that is 10 times smaller [*i.e.*, 12·3], 100 times smaller," and so on.
"If 234 gives the number of decimetres, put below tablets that represent the same length (*a*) in metres [23·4 m.], (*b*) in centimetres [2340 cm.]."
"By what number must I multiply 2 cm. in order to get 20 cm., 4 dm., 2 m.?"
"By what number must I multiply 1·2 dm. to get 12 dm., 12 m.?"
"By what number must I divide 1 m. to get ·01 m., 1 cm., 1 dm.?" and so on.

(4) **Connexion between British and Metric Units of Length.** The children should find out themselves that

4 inches = 1 decimetre.
1 inch = 2·5 centimetres.

Also that the metre is about 39·4 in., and that 22 yds. (1 chain), the length of a cricket-pitch, measures 20 metres.

Section III. Metric Measures of Area and Volume

(1) **Area.** Square centimetres and decimetres are cut out of squared paper. By measuring the square decimetre with the square centimetres the children will find that

100 square centimetres = 1 square decimetre.

A square metre ruled out on the floor or on brown paper could be divided into square decimetres and the relative sizes of the three units be compared.

Square inches might also be introduced—for instance, the fact that 16 square inches = 1 square decimetre is easy to demonstrate.

(2) **Volume.** Fig. 212 gives a plan of a box which the children should make in paper or thin cardboard.

The first box might be one cubic inch, as this is an easy size to measure and make. Then a tiny box of one cubic centimetre and a larger one of one cubic decimetre might

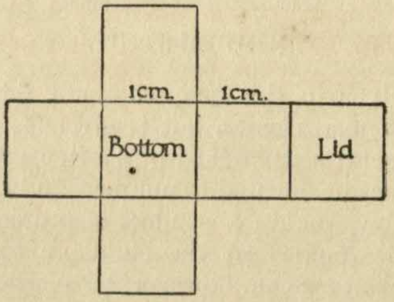


Fig. 212. Plan of Cubical Box

follow. The name ' litre ' should be told to the children, and they should use a standard litre liquid measure.

Wooden models of cubic centimetres and decimetres can be bought, and they are very valuable if shown to the children *after* they themselves have made the boxes.

CHAPTER XII

NUMBER-STUDY

The matter given in this chapter is not intended to form the subject for long classes, but is given as suggestions for drill. The mental arithmetic which often precedes a lesson might sometimes be devoted to number-study. A very short period every day would do wonders in promoting speed and accuracy, and would feed the child's natural interest in numbers. This work can, however, be overdone ; we do not aim at teaching mathematical acrobatic feats, but at cultivating a certain 'at-homeness' with the number-series and an interest in the properties of numbers. The following passage is suggestive :

> Again, children of this age [nine or ten] or even younger can easily be taught to count by twos, threes, up even to nines ; to proceed backwards or forwards, to square each digit, to extract every square and cube root found within 100, to factor and split numbers into fractional parts, and to juggle with number puzzles and play with various tables, magic squares, etc. Very brief periods of intensive drill in this do the work, save time, cultivate the power of attention, keep the mind from lapsing to a slow old Dobbin pace which characterizes so much number work in the schools. The characteristic 'sum' in our primary arithmetic makes much ado about a single one of the steps in the first of the above exercises and thus, by trying to clarify, muddles. The counting instinct is strangulated, treated as an evil to be eliminated, instead of full of the promise and potency of all the higher arithmetic and algebra. Thus, all mental number work should be done with the very utmost rapidity and stop at the first sign of fatigue.[1]

Puzzles and games have not been dealt with, as there are so many excellent books[2] on them and the subjects are such very wide ones.

[1] G. S. Hall, *Educational Problems*, vol. ii (Appleton).

[2] For instance, *Mathematical Recreations and Essays*, by W. W. Rouse Ball (Macmillan).

Section I. Chart-work

(1) **The Number-chart.** Many children have difficulty in realizing the sequence of numbers. They cannot locate a number in the series; for instance, they do not *immediately* realize on hearing or seeing 46 that it lies about half-way between 40 and 50. On examining students as to their visualization of numbers it was found that an astonishing number mentally followed the figures in a straight line (either horizontal or, more usually, inclined at an angle of about 30°) up to 20. Then there was a break, and a second line was begun below the first, but extending indefinitely beyond it. Others followed zigzag lines.

It seems, therefore, extremely important to give children very early in their arithmetical training some form and arrangement of numbers that may enable them to visualize and organize the number-series in their minds. Number-study is in itself very interesting, and children delight in finding out symmetrical arrangements such as are shown in Chapter VI.

Every child should have a number-chart extending to 50, 100, or 200. He should be allowed to refer to this chart constantly, and should use it to verify the answers to his sums. It does not really signify whether the chart used has the numbers 1–10 in a vertical line, as in Fig. 213, or in a horizontal line, as shown in Chapter VI. One form or the other should be chosen and *adhered to.* That shown in Fig. 213 has the advantage of forming a more conveniently shaped wall-chart if extended to 200 or 300. If a large wall-chart is made of wood, drawing-pin hooks may be stuck into it and all sorts of interesting exercises in weights, measures, and money be performed on it. The following are some suggestions for exercises on such a chart.

(*a*) *Building up the Chart.* It is useful to let the children build up the chart themselves, not simply by writing the numbers in sequence, but by filling in various columns of numbers in an order designed to teach salient connexions. A chart made by a child is seldom sufficiently well done to be desirable for reference. In any case, a printed one is better for this purpose, as the neatness and evenness of the figures

and spacing are training in themselves, and make visualization easier.

The children having been provided with squared paper, the figures might be filled in according to the following directions :

" Take a red pencil. Write figure **1** in the square in the top left-hand corner. Fill in figures **2** to **10** in a vertical

1	11	21	31	41	51	61	71	81	91	101	111	121	131	141	151
2	12	22	32	42	52	62	72	82	92	102	112	122	132	142	152
3	13	23	33	43	53	63	73	83	93	103	113	123	133	143	153
4	14	24	34	44	54	64	74	84	94	104	114	124	134	144	154
5	15	25	35	45	55	65	75	85	95	105	115	125	135	145	155
6	16	26	36	46	56	66	76	86	96	106	116	126	136	146	156
7	17	27	37	47	57	67	77	87	97	107	117	127	137	147	157
8	18	28	38	48	58	68	78	88	98	108	118	128	138	148	158
9	19	29	39	49	59	69	79	89	99	109	119	129	139	149	159
10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160

Fig. 213[1]

See also Fig. 147 for an alternative arrangement.

column below the 1. What number will come in the second column beside the 1 ? beside the 5 ? beside the 10 ? Fill in these three numbers. Fill in the whole of the top or 1 row. Fill in the whole of the bottom or 10 row. Fill in the whole of the 5 row. Take a black pencil and fill in these numbers —**19, 14, 12, 16, 39, 46, 54, 62, 82, 64, 56, 49. 23, 33, 43**; complete this row. **17, 47, 57, 87**; complete this row. **18, 38, 98, 68**; complete this row. Complete the chart."

When the children have had some practice with the chart they may be required to make one (with or without the help of the wall-chart) in one of the following ways :

(i) Make a chart to 100, writing down only the numbers divisible by 2 (or by 3 or 5).

(ii) Make a chart to 100, filling in only the multiples of ten (in red) and the prime numbers (in black).

[1] See *The Psychology of Number*, p. 188, for a similar chart with columns having figures 0–9, 10–19, and so on.

(*b*) *Addition and Subtraction.* Small children should use a chart with larger squares and represent their sums in reversible counters—*e.g.*, in the sum 8 + 9 the child would put out 8 counters with red side up and 9 with black side up. In 17 - 9, 17 black counters are put out, then 9 are turned to the red side, *beginning with the one on square* 17.

When a child has got accustomed to the chart he should put fewer counters out. For instance, in 43 + 39 he would put a counter on square 43 (and if desired on 73 or 52) and on 82.

In the same way, in the subtraction 82 - 43 a counter would be put only on 82 (if desired also on 42 or 79) and on 39. Here, if enough practice has not been given, the child will make the mistake of not counting as subtrahend the square on which the counter is. However, this mistake is easily remedied.

Drill should be given in adding and subtracting in 10*s*., 5*s*., 9*s*., 11*s*., 20*s*., 8*s*., 15*s*., 19*s*., etc.

(*c*) *Multiplication and Division.* Tables may be built up with reversible counters—*e.g.*, 3 red, 3 black, 3 red, and so on until the table of threes is completed.

Division by grouping can also be done in exactly the same way. " How many threes are there in 18 ? " would lead to the child's putting out 3 red, 3 black counters, etc., until 18 was reached.

For the ' sharing ' aspect of division the chart is not so clear. In the example " 18 apples shared among 3 children " counters of three colours should be used, and finally the number of each colour counted, or else the child must say " One to A, one to B, one to C," putting out red counters, " Two to A, two to B, two to C," adding three black counters, and so on.

Examples such as the following are also instructive :

" Arrange counters on the squares 1 to 20 to show clearly that $10 \times 2 = 2 \times 10$."

" Arrange counters on the squares 1 to 48 to show that $4 \times 12 = 8 \times 6 = 12 \times 4 = 48$."

(*d*) *Money and Weights and Measures.* The different money tables may be built up on the chart. For instance, all

multiples of 12 may be coloured red, or a cardboard shilling may be stuck on the wall-board. In this way the child learns to tell at a glance how to reduce pence to shillings and *vice versa.* Other signs might be used for ounces and pounds.

1	2	3		5		7			
11		13				17		19	
		23						29	
31						37			
41		43				47			
		53						59	
61						67			
71		73						79	
		83						89	
						97			
101		103				107		109	
		113							
						127			
131						137		139	
								149	
151						157			
		163				167			
		173						179	
181									
191		193				197		199	

Fig. 214. PRIME NUMBERS

(*e*) *Prime Numbers* (see Fig. 214). Every child of nine should know all the prime numbers below a hundred. "Progress" Game 1 gives the following method of finding them on the chart:

"Cover, with black side up, all numbers divisible by 2 except 2 itself.

"Cover, with red side up, all numbers divisible by 3 except 3 itself. (Do not put a second counter on any square already covered.) Cover, with red side up, all numbers divisible by 5 except 5 itself. Cover, with red side up, all numbers divisible by 7 except 7 itself. When this has been done the following numbers will be left uncovered: 1, 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97. These are the prime numbers."

When this has been done the child should write out the prime numbers, and refer to his list when necessary. He should not be required to memorize them all at once, but should try to cover them up on the blank chart and then by checking his results *gradually* learn them. Should he leave a number—for instance, 87—uncovered, he should be required

to factorize it out in full—87 = 3 × 29—so as not to make the same mistake again.

Questions of the following type will help to fix the prime numbers:

"How many prime numbers are there between 1 and 20; 20 and 40; 40 and 60; 60 and 80; 80 and 100?"

"Write out in four columns the prime numbers ending in 1, 3, 7, and 9."

"Write out the numbers that are not prime, but which end in 1, 3, 7, and 9. Factorize them thus—27 = 3 × 9. How many have 3 as a factor?"

In "Progress" Game 1 questions are printed on the card for the child to use. There is also a teacher's card.

(*f*) *Factors and Multiples.* The following suggestions are taken from "Progress" Game 1. The teacher names a number and lets the children cover up the factors of it. For instance, 36 is called, and they cover up 1, 2, 3, 4, 6, 9, 12, 18. This may be continued until only the prime numbers above 50 are left. The exercise might be varied by making the children write down each equation—*e.g.*, 36 = 36 × 1 = 18 × 2 = 12 × 3, and so on.

A prime number below 50 is named, and the children cover up its multiples. For instance, 23 is called, and 46, 69, and 92 are covered up.

(2) **The Multiplication Chart.** This is shown in Fig. 111. It is an excellent summary of the multiplication tables, and a study of it helps a child to remember his tables and their connexion with one another.

This chart may be used for drill in factors, square roots, and squaring. For instance, when a number—let us say, 28—is named the children will find by reference to the chart that it occurs twice—viz., 4 × 7 and 7 × 4. 36 occurs five times—viz., as 3 × 12, 4 × 9, 6 × 6, 9 × 4, and 12 × 3. If "Progress" Games 2 or 3 be used, the tickets bearing the products may be drawn and put on the chart in any order. A very good blackboard game may be organized on these lines if the teacher prepares tickets with the products, which may be either dealt out to the children or be drawn by them.

Practice should be given in reducing to the lowest factors.

For instance, 84 would first be given as 12×7 or 7×12 from the chart, the 12 would then have to be broken up, and the result $2 \times 2 \times 3 \times 7 = 84$ obtained. The signs 2^2 and $\sqrt{4}$ may quite well be given to children of nine once they have understood multiplication. It will be noticed that the squares of numbers 1 to 12 form a diagonal on the chart. The child should be required to find and then write out the squares of all numbers 1 to 12.

Questions of the following types will help to make the child familiar with the chart:

" Pick out all numbers ending in 7. Factorize them [7, 27, and 77 only]. Pick out all numbers ending in 1 [1, 21, 81, 121]. Which of these occur only once ? "

" Which tables have the greatest variety of figures in the units place ? Which table has only one figure all through in the units place ? Which has two ? Which five ? "

" Write down the numbers which can only be resolved into two factors [*e.g.*, $55 = 11 \times 5$]."

For further suggestions see " Progress " Games 2 and 3.

Section II. Factors and Square Root

(1) **Factors.** Children love factorizing, and some practice in it is essential. Once they have found the prime numbers below 20 they should be required to factorize in an orderly manner. If 924 were the number to be factorized, the child should see at a glance that it is divisible by 2 or by 4. Next he should try 3 and then 7 (5 being put out of the question at once).

No elaborate rules should be given as to divisibility. At first the child should learn to distinguish between odd and even numbers. Then divisibility by 10 and 5 is easily realized once the tables have been built up and the number-chart used.

The test for divisibility by 3 or 9 is useful later on, but should not be taught too soon. The following types of exercises are recommended :

Drill on special numbers of frequent occurrence in arith-

metical calculations—*e.g.*, 100, 112, 1760, 4840, and so on. Figs. 215, 216, and 217 show possible drill-cards. They

112

112 ÷ 2 =	112 × 20 =
112 ÷ 4 =	$\frac{1}{4}$ of 112 =
$\frac{1}{7}$ of 112 =	$\frac{1}{4}$ of 2240 =
112 ÷ 16 =	$\frac{1}{20}$ of 2240 =
112 ÷ 28 =	28 × 4 =
14 × = 112	2240 ÷ 20 =

Fig. 215. Drill-card on 112

1760

1760 = 176 ×	1760 ÷ 5 =
1760 = 110 ×	1760 ÷ 11 =
1760 = 40 ×	1760 ÷ 60 =
1760 = 11 ×	1760 ÷ 44 =
1760 = 55 ×	$\frac{1}{4}$ of 1760 =
1760 = 88 ×	$\frac{1}{8}$ of 1760 =

Fig. 216. Drill-card on 1760

100 = 2 ×	$\frac{1}{2}$ of 100 =
100 = 4 ×	$\frac{1}{4}$ of 100 =
100 = 5 ×	$\frac{1}{5}$ of 100 =
100 = 10 ×	$\frac{1}{10}$ of 100 =
50 = 2 ×	$\frac{1}{20}$ of 100 =
75 = 3 ×	$\frac{3}{4}$ of 100 =

Continued on 200, 300, up to 1000

Fig. 217. Drill-card on 100

should be worked through at high pressure, and there should be no dawdling over them.

For the younger children an exercise in changing a multiplication sum into one in division is desirable. Fig. 133

shows a card in which the numbers 24 and 18 are worked on in this way.

Children may also be given a number—let us say, 54—and be required to make as many equations with it as possible in a given time—*e.g.*,

$$\begin{aligned} 54 &= 2 \times 27. & \tfrac{1}{2} \text{ of } 54 &= 27. \\ &= 9 \times 6. & \tfrac{1}{6} \text{ of } 54 &= 9. \\ &= 3 \times 18. & \tfrac{1}{3} \text{ of } 54 &= 18. \end{aligned}$$

Or questions may be set—*e.g.*, "Find two factors of 54 which add up to 21." "What fraction of 54 equals 9 ?" "What is the

18	24	15	33	27
6	50	35	30	36
25	21	42	14	10
24	40	28	20	3
9	27	45	36	12

Fig. 218

Cover up with red counters all numbers divisible by 7 ; with yellow counters all those divisible by 5 and not already covered ; and with blue counters those divisible by 3 and not already covered.

largest number of factors into which 54 may be resolved ?" "54*d*. are how many shillings ?" "If I divide 54 by 6, by what number must I multiply the answer in order to get 36 ?"

Fig. 218 shows a very easy exercise in factorizing. A pattern is formed if the squares are covered correctly.

(2) **Squares and Square Root.** On the multiplication chart it will be noticed that the squares of the numbers form a diagonal—1, 4, 9, 16, and so on. The children might be invited to extend the table to 20. In any case, they should know the squares of all numbers up to 20. These are given in Fig. 219, and are interesting to study in a series. Note the central 5^2 and the symmetrical occurrence of the figures in the units place above and below it.

The children should be led to connect square root with a square figure. The following is a suggestive exercise :

"Draw rectangular figures with area of (*a*) 16 sq. in.,

(*b*) 25 sq. in. Which of the three figures in (*a*) has the greatest perimeter, and which has the shortest ? "

To find the square root of a number the children should resolve it into factors, thus :

$$
\begin{aligned}
\sqrt{324} &= \sqrt{2 \times 162} \\
&= \sqrt{2 \times 2 \times 81} \\
&= \sqrt{2 \times 2 \times 9 \times 9} \\
&= 2 \times 9 \\
&= 18.
\end{aligned}
$$

Though in this case it would be waste of time to bring to prime factors, it will often be necessary for a child to do so.

$1^2 = 1$	$11^2 = 121$
$2^2 = 4$	$12^2 = 144$
$3^2 = 9$	$13^2 = 169$
$4^2 = 16$	$14^2 = 196$
$5^2 = 25$	$15^2 = 225$
$6^2 = 36$	$16^2 = 256$
$7^2 = 49$	$17^2 = 289$
$8^2 = 64$	$18^2 = 324$
$9^2 = 81$	$19^2 = 361$
$10^2 = 100$	$20^2 = 400$

Fig. 219. SQUARES

These results should be kept for reference and gradually be memorized.

$1^3 = 1$	$11^3 = 1331$
$2^3 = 8$	$12^3 = 1728$
$3^3 = 27$	$13^3 = 2197$
$4^3 = 64$	$14^3 = 2744$
$5^3 = 125$	$15^3 = 3375$
$6^3 = 216$	$16^3 = 4096$
$7^3 = 343$	$17^3 = 4913$
$8^3 = 512$	$18^3 = 5832$
$9^3 = 729$	$19^3 = 6859$
$10^3 = 1000$	$20^3 = 8000$

Fig. 220. CUBES—VALUABLE AS AN EXERCISE IN MULTIPLICATION

(3) **Cubes and Cube Root.** These need not detain us long. The children should know what ' cube ' and ' cube root ' mean. The idea should be associated with volume, just as square root may be connected with area. They should find the cubes of numbers 1 to 10, but should not be required to learn them.

Fig. 220 gives these cubes. A child requiring practice in multiplication might be asked to complete a series to 20 in order, for instance, to see if it is a fact that the corresponding figures in the units place above and below the central 5^3 always add up to 10—*e.g.*, 4 + 6, 7 + 3, 8 + 2, 9 + 1.

(4) **Dr Montessori's Squares and Cubes.** Dr Montessori has a set of squares and cubes made out of short bead-bars. By placing three 3-bars together the child would get a square of 3×3, containing 9 beads. By placing three such squares on top of one another he would get a cube of $3 \times 3 \times 3$, containing 27 beads. The unfortunate part of Dr Montessori's squares and cubes is that they cannot be undone—they are permanently wired.

Cubes may be built up with the 100-rod chains described on p. 79. These have the advantage of being also useful for calculations on volume.

Section III. Highest Common Factor and Least Common Multiple

(1) **Highest Common Factor.** No difficulty should be experienced in teaching highest common factor or least common multiple. Children usually like this part of their work. Only simple examples should be given, and they should be worked out by straightforward factorizing.

The ideas of highest common factor and least common multiple may be introduced by simple problems, but it must be borne in mind that problems based on either are comparatively rare—those given in text-books are often far-fetched and unreal. Highest common factor and least common multiple must be taught as useful *means*, but too much time must not be spent over elaborate problems. The following are a few simple ones which might serve to introduce the subjects :

"Here are two pieces of paper. One is 12 and the other 18 in. long. I want to cut them up so that I shall get pieces exactly the same length and as long as possible. Into how many pieces shall I cut each, and how long will my final pieces be?"

"I have 20 counters and you have 32. Let us each divide ours into piles so that all the piles will contain the same number of counters."

"What is the largest number that will go exactly into 48 and 54?"

In *Fundamental Arithmetic*, Part III, the following direct method is used for explaining highest common factor and least common multiple, and it seems clearer than introducing problems first:

2	3	6	8	12	16	18	20	24	25	30
		3	4	6	8	9	10	12	5	15
		2	2	4	4	6	5	8		10
				3	2	3	4	6		6
				2		2	2	4		5
								3		3
								2		2

Below each of the numbers in the top row its factors have been arranged in a column. Since 1 will divide every number exactly it is left out of account in dealing with factors.

(i) What are the *common factors* of 12 and 20 ?
Answer, 2 and 4.

(ii) What is the *highest common factor* of 12 and 20 ?
Answer, 4.

(iii) What are common factors of 6 and 25 ?
Answer, None.

(iv) What is the highest common factor of 3 and 6 ?
Answer, 3.

The next step would be to give examples to be worked by straightforward factorizing—*e.g.*, " Find the highest common factor of 18, 30, and 42."

$$\begin{aligned} 18 &= 2 \times 3 \times 3. \\ 30 &= 2 \times 3 \times 5. \\ 42 &= 2 \times 3 \times 7. \\ \text{H.C.F.} &= 2 \times 3 \\ &= 6. \end{aligned}$$

The advantage of knowing how to find the highest common factor of two or more numbers should be made clear by exercises in simplifying or cancelling fractions.

(2) **Least Common Multiple.** (*a*) The following are a few very simple problems by which the least common multiple might be introduced, though here again the direct method with numbers only seems best.

" What is the smallest sum of money that can be paid exactly (i) in threepenny-, sixpenny-, and shilling pieces, (ii) in half-crowns and florins ? "

"What is the shortest length of ribbon which I can cut up exactly into 6-in. or 8-in. lengths?"

(*b*) The least common multiple can be clearly explained by using the multiplication table chart. The following questions will help to make clear the terms multiple and least common multiple:

(i) "Look along the 2 and 3 row, and pick out all the numbers that occur in both." *Answer*, 6, 9, 12, 18, 24. "These are common multiples of 2 and 3. The least common multiple of 2 and 3 is 6."

(ii) "Pick out the common multiples of 3 and 4." *Answer*, 12, 24, 36, 48.

(iii) "Find three common multiples of 3, 4, and 6." *Answer*, 12, 24, 36.

(iv) "Find the least common multiple of 6 and 8; 3, 5, and 6; 9 and 6." *Answer*, 24; 30; 18.

After exercises of this kind simple examples should be given of finding the least common multiple by straightforward factorizing—*e.g.*, "Find the least common multiple of 24, 36, and 42."

$$24 = 2 \times 2 \times 2 \times 3 = 2^3 \times 3.$$
$$36 = 2 \times 2 \times 3 \times 3 = 2^2 \times 3^2.$$
$$42 = 2 \times 3 \times 7 = 2 \times 3 \times 7.$$
$$\text{L.C.M.} = 2^3 \times 3^2 \times 7.$$

APPENDIX I

APPARATUS

All accessories for the "Welbent" Scheme can be obtained from Messrs E. J. Arnold and Son, Ltd.; Number Charts 1–120 *from Messrs Philip and Tacey.*

Long and Short Bead-bars (E. J. Arnold).
Number-rods in wood, "The County" set (Philip and Tacey).
All-about-ten Number-cards, Set 2 (pips arranged in fives) strongly recommended (Philip and Tacey).
Adaptable Counters, $\frac{3}{4}$-inch or 1-inch diameter. Two colours only—*e.g.*, black or blue one side, red or green the other (E.S.A. or Philip and Tacey).
Digit-tablets, boxes A or B (E. J. Arnold).
Stout Sticks, 2 to $2\frac{1}{2}$ inches. Easily dyed in two colours. Packets of ten glued together and some loose sticks to make into bundles with elastic bands. Blocks of wood for hundreds.
Or *Strong Cardboard*. Square decimetre for hundreds; strips of ten square centimetres for tens; square centimetre for units.
Cardboard Coins (Philip and Tacey, E. J. Arnold, or E.S.A.).
"*Simple Form Insets*," squares and circles (Philip and Tacey).
Geometrical Templates in cardboard (E. J. Arnold or E.S.A.).
Cut-out Wooden Figures (Philip and Tacey or E.S.A.).

The Holborn Table Book will be found very useful for individual children for memorizing the tables. It is well illustrated.

At present there is little apparatus on the market, but wooden cubes, beads, etc., may be bought in toy-shops—and much can be done with waste materials.

APPENDIX II

SYLLABUS ARRANGED IN STAGES

STAGE I, *Pink*[1]

Tracing on sandpaper or embossed figures 1 to 6 inclusive.
Recognition of number-groups and figures 1 to 6 inclusive.
Plus, minus, and equal signs.
Building equations with number-rods, bead-bars, and counters.
Representing given equations in concrete form and solving them.
Sorting and counting objects.
Counting to 20 or 30.

STAGE II, *Blue*

Tracing on sandpaper or embossed figures 6 to 10 inclusive.
Recognition of number-groups and figures up to 10 inclusive.
Building up equations with number-rods, bead-bars, or counters.
Equations involving all combinations in addition and subtraction involving numbers up to 10.
Counting to 30 or 50.

STAGE III, *Buff*

Building up equations involving numbers 1 to 20.
Place-value of tens and units.
Counting exercises on the 30-bead bar.
Equations involving addition and subtraction of numbers below 20 (composition of numbers up to 20).
The meaning of the multiplication sign. Interpreting, for example, 2×3 concretely, and expressing repeated addition as multiplication.
The penny, halfpenny, and farthing.
The fact that twelve pennies make one shilling.
Very simple problems.

STAGE IV, *Green*

Building up the tables of twos and threes.
Building up equations in addition and subtraction with bead-bars and number-rods.

[1] The exercise-cards for each stage should be of a distinctive colour. The colours given in this scheme are those used in the "Welbent" Series. This series covers the syllabus to the end of Stage 5.

Equations involving addition, subtraction, and multiplication with numbers 1 to 30.

Simple problems in the above rules.

Simple money calculations to about 2*s*. 6*d*.

Sharing among two, three, or four.

Grouping in twos, threes, and fours.

The division sign.

Counting to 60 in tens and fives

Stage V, *Mauve*

Exercises with the 60-chain or bar. Building up equations and solving set equations.

Building up the tables of tens and fives, using the 60-bar.

Building up the table of fours (with short bead-bars).

Building up equations in division.

Easy equations in multiplication and division.

More difficult addition and subtraction.

Simple problems in all four rules.

Halves and quarters.

Inches and feet. Measuring.

The ounce and the pound.

Shopping activities and problems involving sums below 5*s*.

Stage VI, *Yellow (age* 7 *years)*

Exercises with the 100-chain. Building up and working equations.

Place-value : hundreds, tens, and units.

Exercises with notation-cards and bead-bars.

Building up the table of sixes and eights.

Addition sums (vertical), with ' carrying.'

Subtraction sums (vertical), with the ' difficulty.'

Multiplication with 10 as multiplier.

Multiplication with multipliers below six.

Division with divisors 2, 3, or 4.

Problems involving multiplication and division.

Exercises with money board—addition and subtraction up to £1.

Shopping exercises involving yards, pints, and quarts.

Measuring in inches, feet, and yards, also $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{1}{4}$ inches.

Stage VII, *Red* (7½ *years*)[1]

Rapid addition and subtraction to 100 in equation form.

Exercises on the 100-chart ; even and odd numbers, divisibility by 3, 5, 10, etc.

[1] The child is now ready for a text-book such as *Fundamental Arithmetic*, by P. B. Ballard (London University Press).

Building up the table of nines ; chart.

Notation ; reading figures and writing numbers to dictation up to 1000.

Exercises with 1000-chain.

Addition (vertical form) involving numbers up to 1000.

Subtraction (vertical form) involving numbers up to 1000.

Addition and subtraction of money (shillings, pence, farthings).

Easy multiplication and division of money.

Shopping problems.

Problems involving two processes, giving practice in classification, and requiring knowledge of days of the week and months.

The above is intended to represent what has been termed the " working syllabus " (see Introduction to this book, p. 6).

Many teachers will find it profitable to introduce the child to the 100-chain in Stage IV or V, connecting its composition with that of a 10-bar. Even when this is done the actual test work might follow this syllabus.

BIBLIOGRAPHY

BRANFORD, BENCHARA: *A Study of Mathematical Education* (Clarendon Press).

BRIDEOAKE, E., AND I. GROVES: *Arithmetic in Action* (University of London Press).

CAJORI, FLORIAN: *A History of Mathematics*, second edition (Macmillan, 1938).

Craftsmanship in the Teaching of Arithmetic and Elementary Mathematics (Blackie).

CUNNINGTON, S.: *The Story of Arithmetic: A Short History of its Origin and Development* (Allen and Unwin).

DRUMMOND, MARGARET: *Learning Arithmetic by the Montessori Method* (Harrap).

DRUMMOND, MARGARET: *The Psychology and Teaching of Number* (Harrap).

HEMMINGS, F. J.: *The Teaching of Arithmetic and Elementary Mathematics* (Blackie).

HOGBEN, LANCELOT: *Mathematics for the Million* (Allen and Unwin).

KENWICK, E.: *Number in the Nursery and Infant School* (Kegan Paul).

MCLELLAN, J., AND J. DEWEY: *The Psychology of Number* (Appleton).

MONTESSORI, MARIA: *The Advanced Montessori Method* (Heinemann).

MONTESSORI, MARIA: *The Montessori Method* (Heinemann).

POTTER, F. F.: *The Teaching of Arithmetic* (Pitman).

PUNNETT, MARGARET: *The Groundwork of Arithmetic* (Longmans, Green and Co.).

SCHONELL, F.: *Diagnosis of Individual Difficulties in Arithmetic* (Oliver and Boyd).

SMITH, D. E.: *The Teaching of Elementary Mathematics* (Macmillan Co., New York).

SUMNER, W.: *The Teaching of Arithmetic and Elementary Mathematics* (Blackwell).

THOMSON, J. B.: *The Art of Teaching Arithmetic* (Longmans, Green and Co.).